

Fuzzballs with internal excitations

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ABSTRACT: We construct general 2-charge D1-D5 horizon-free non-singular solutions of IIB supergravity on T^4 and $K3$ describing fuzzballs with excitations in the internal manifold; these excitations are characterized by arbitrary curves. The solutions are obtained via dualities from F1-P solutions of heterotic and type IIB on T^4 for the $K3$ and T^4 cases, respectively. We compute the holographic data encoded in these solutions, and show that the internal excitations are captured by vevs of chiral primaries associated with the middle cohomology of T^4 or $K3$. We argue that each geometry is dual to a specific superposition of R ground states determined in terms of the Fourier coefficients of the curves defining the supergravity solution. We compute vevs of chiral primaries associated with the middle cohomology and show that they indeed acquire vevs in the superpositions corresponding to fuzzballs with internal excitations, in accordance with the holographic results. We also address the question of whether the fuzzball program can be implemented consistently within supergravity.

KEYWORDS: AdS-CFT Correspondence, Black Holes in String Theory, Gauge-gravity correspondence.

Contents

1. Introduction	2
2. Fuzzball solutions on T^4	6
2.1 Chiral null models	6
2.2 The IIA F1-NS5 system	7
2.3 Dualizing further to the D1-D5 system	9
3. Fuzzball solutions on $K3$	10
3.1 Heterotic chiral model in 10 dimensions	10
3.2 Compactification on T^4	11
3.3 String-string duality to P-NS5 (IIA) on $K3$	12
3.4 T-duality to F1-NS5 (IIB) on $K3$	13
3.5 S-duality to D1-D5 on $K3$	16
4. D1-D5 fuzzball solutions	17
5. Vevs for the fuzzball solutions	21
5.1 Holographic relations for vevs	22
5.2 Application to the fuzzball solutions	25
6. Properties of fuzzball solutions	27
6.1 Dual field theory	27
6.2 Correspondence between geometries and ground states	28
6.3 Matching with the holographic vevs	30
6.4 A simple example	31
6.5 Selection rules for curve frequencies	34
6.6 Fuzzballs with no transverse excitations	35
7. Implications for the fuzzball program	36
A. Conventions	39
A.1 Field equations	39
A.2 Duality rules	41
B. Reduction of type IIB solutions on $K3$	42
B.1 S-duality in 6 dimensions	45
B.2 Basis change matrices	46
C. Properties of spherical harmonics	47
D. Interpretation of winding modes	47

1. Introduction

Over the last few years an interesting new proposal for the gravitational nature of black hole microstates has emerged [1–3]; see also [4–6], and [7]. According to this proposal there should exist non-singular horizon-free geometries associated with the black hole microstates. These so-called fuzzball geometries should asymptotically approach the original black hole geometry and should generically differ from each other around the horizon scale. In this scenario the black hole provides only an average statistical description of the physics and thus longstanding issues such as the information loss paradox would be resolved. The underlying physics of the black hole would not be conceptually different from that of a star, with the temperature and entropy being of statistical origin. Given the importance of understanding black hole physics and its implications for quantum gravity, this proposal should be developed, explored and tested where possible.

Many issues need to be addressed to implement the fuzzball proposal at a quantitative and precise level. The proposal requires the existence of exponential numbers of horizon-free non-singular solutions for each black hole. So the most basic of questions is whether one can find such a number of solutions with the required properties and moreover what precisely are the required properties for any given black hole. Moreover one would like to show quantitatively how black hole properties emerge upon coarse-graining; for this one needs to know the precise relationship between geometries and microstates.

Much of the recent work on this proposal has been focused on constructing fuzzball geometries for certain supersymmetric black holes with macroscopic horizons; for a summary of progress in this direction see [8]. The method of construction here uses crucially supersymmetry and known classifications of supersymmetric solutions: one looks for non-singular horizon-free supersymmetric solutions with the correct charges to match those of the black hole.

This method however has a number of limitations. One is that the supersymmetric classifications are not sufficiently restrictive for cases of interest and thus one needs a specific ansatz to make progress. To date many of the fuzzball geometries constructed are rather atypical; for example, they have angular momenta much larger than those of a typical black hole microstate. Whilst families of typical geometries are presumably contained in the supersymmetric classification, finding an ansatz to construct families rather than isolated examples is not easy.

Another key issue is that one does not know precisely what is the relationship between a given geometry and the black hole microstates. This in turn means that one does not know whether one has constructed the correct geometries to describe the black hole. Nor does one know whether one has enough geometries to account for the black hole entropy upon geometric quantization, using the methods of [9]. For example, in cases where the

dual theory has distinct Higgs and Coulomb branches, one needs to determine whether a given fuzzball geometry describes Higgs or Coulomb branch physics. More importantly, one would like to see explicitly how black hole properties emerge upon coarse graining; to understand how to do such a computation properly the precise relation between the fuzzball geometries and microstates is crucial.

To address this issue, we have advocated and developed the use of AdS/CFT methods. That is, the supersymmetric black holes of interest admit a dual CFT description and the fuzzball geometries therefore have a decoupling limit which is asymptotically AdS. One can therefore use well-developed techniques of AdS/CFT, in particular Kaluza-Klein holography [10], to extract field theory data from the geometry and diagnose precisely what the geometry describes.

It is worth emphasizing at this point that the AdS/CFT correspondence both motivates and supports the fuzzball picture. The gravity/gauge theory dictionary relates a given asymptotically AdS geometry to either a deformation of the CFT or the CFT in a non-trivial vacuum characterized by the expectation values of gauge invariant operators. Conversely, one expects that for any stable state of the CFT (such as the BPS states) there exists an asymptotically AdS solution, whose asymptotics encode the vevs of gauge invariant operators in that state. If the field theory is in a pure state, there is no entropy and one does not expect the corresponding geometry to have a horizon, and hence entropy. AdS/CFT thus implies that the field theory in a given pure stable (black hole) state should have a geometric dual with no horizon; there is however no guarantee that the geometry should be well-described by supergravity alone, i.e. weakly curved everywhere.¹

In our recent papers [11, 12], we have discussed in some detail the case of the D1-D5 system, for which (some) fuzzball geometries were constructed in [1]. Since this is a 2-charge system, there is no macroscopic horizon: the naive geometry is singular, with the horizon believed to form on taking into account α' corrections. Whilst this is not a macroscopic black hole system, there are a number of reasons to explore this case fully before moving on to supersymmetric macroscopic black holes.

Firstly, one can obtain *all* fuzzball geometries in this system by dualities from known solitonic solutions of F1-P systems. Thus one should be able to account for all the entropy, and show how the average black hole description emerges. Moreover, the dual description of this black hole is the simplest and best understood: the black hole entropy arises from the degeneracy of the Ramond ground states of the dual (4, 4) CFT. This is an ideal system in which to address the question of what is the precise correspondence between geometries and microstates, and moreover how the properties of given microstates determine and characterize the fuzzball geometries.

In the original work of [1], only a subset of the 2-charge fuzzball geometries were constructed using dualities from F1-P solutions. Recall that the D1-D5 system on T^4 is related by dualities to the type II F1-P system, also on T^4 , whilst the D1-D5 system on

¹There are additional subtleties in low dimensional quantum field theories due to the strong infrared fluctuations. More properly one should view a given fuzzball geometry as dual to a wavefunction on the Higgs branch of the field theory, but it seems in any case likely that such wavefunctions would be localized around specific regions in the large N limit and thus that this issue does not play a key role at infinite N .

$K3$ is related to the heterotic F1-P system on T^4 ; the exact duality chains needed will be reviewed in sections 2 and 3. Now the solution for a fundamental string carrying momentum in type II is characterized by 12 arbitrary curves, eight associated with transverse bosonic excitations and four associated with the bosonization of eight fermionic excitations on the string [13]. The corresponding heterotic string solution is characterized by 24 arbitrary curves, eight associated with transverse bosonic excitations and 16 associated with charge waves on the string.

In the work of [1], the duality chain was carried out for type II F1-P solutions on T^4 for which only bosonic excitations in the transverse R^4 are excited. That is, the solutions are characterized by only four arbitrary curves; in the dual D1-D5 solutions these four curves characterize the blow-up of the branes, which in the naive solutions are sitting in the origin of the transverse R^4 , into a supertube. In this paper we carry out the dualities for generic F1-P solutions in both the T^4 and $K3$ cases, to obtain generic 2-charge fuzzball solutions with internal excitations. Note that partial results for the T^4 case were previously given in the appendix of [3]; we will comment on the relation between our solutions and theirs in section 2. The general solutions are then characterized by arbitrary curves capturing excitations along the compact manifold M^4 , along with the four curves describing the blow-up in R^4 . They describe a bound state of D1 and D5-branes, wrapped on the compact manifold M^4 , blown up into a rotating supertube in R^4 and with excitations along the part of the D5-branes wrapping the M^4 .

The duality chain that uses string-string duality from heterotic on T^4 to type II on $K3$ provides a route for obtaining fuzzball solutions that has not been fully explored. One of the results in this paper is to make explicit all steps in this duality route. In particular, we work out the reduction of type IIB on $K3$ and show how S-duality acts in six dimensions. These results may be useful in obtaining fuzzball solution with more charges.

In our previous work [11, 12], we made a precise proposal for the relationship between the 2-charge fuzzball geometries characterized by four curves $F^i(v)$ and superpositions of R ground states: *a given geometry characterized by $F^i(v)$ is dual to a specific superposition of R vacua with the superposition determined by the Fourier coefficients of the curves $F^i(v)$.* In particular, note that only geometries associated with circular curves are dual to a single R ground state (in the usual basis, where the states are eigenstates of the R-charge). This proposal has a straightforward extension to generic 2-charge geometries, which we will spell out in section 6, and the extended proposal passes all kinematical and accessible dynamical tests, just as in [11, 12].

In particular, we extract one point functions for chiral primaries from the asymptotically AdS region of the fuzzball solutions. We find that chiral primaries associated with the middle cohomology of M^4 acquire vevs when there are both internal and transverse excitations; these vevs hence characterize the internal excitations. Moreover, there are selection rules for these vevs, in that the internal and transverse curves must have common frequencies.

These properties of the holographic vevs follow directly from the proposed dual superpositions of ground states. The vevs in these ground states can be derived from three point functions between chiral primaries at the conformal point. Selection rules for the

latter, namely charge conservation and conservation of the number of operators associated with each middle cohomology cycle, lead to precisely the features of the vevs found holographically.

To test the actual values of the kinematically allowed vevs would require information about the three point functions of all chiral primaries which is not currently known and is inaccessible in supergravity. However, as in [12], these vevs are reproduced surprisingly well by simple approximations for the three point functions, which follow from treating the operators as harmonic oscillators. This suggests that the structure of the chiral ring may simplify considerably in the large N limit, and it would be interesting to explore this question further.

An interesting feature of the solutions is that they collapse to the naive geometry when there are internal but no transverse excitations. One can understand this as follows. Geometries with only internal excitations are dual to superpositions of R ground states built from operators associated with the middle cohomology of M^4 . Such operators account for a finite fraction of the entropy, but have zero R charges with respect to the $SO(4)$ R symmetry group. This means that they can only be characterized by the vevs of $SO(4)$ singlet operators but the only such operators visible in supergravity are kinematically prevented from acquiring vevs. Thus it is consistent that in supergravity one could not distinguish between such solutions: one would need to go beyond supergravity to resolve them (by, for instance, considering vevs of singlet operators dual to string states).

This brings us to a recurring question in the fuzzball program: can it be implemented consistently within supergravity? As already mentioned, rigorously testing the proposed correspondence between geometries and superpositions of microstates requires information beyond supergravity. Furthermore, the geometric duals of superpositions with very small or zero R charges are not well-described in supergravity. Even if one has geometries which are smooth supergravity geometries, these may not be distinguishable from each other within supergravity: for example, their vevs may differ only by terms of order $1/N$, which cannot be reliably computed in supergravity.

The question of whether the fuzzball program can be implemented in supergravity could first be phrased in the following way. Can one find a complete basis of fuzzball geometries, each of which is well-described everywhere by supergravity, which are distinguishable from each other within supergravity and which together span the black hole microstates? On general grounds one would expect this not to be possible since many of the microstates carry small quantum numbers. We quantify this discussion in the last section of this paper in the context of both 2-charge and 3-charge systems.

To make progress within supergravity, however, it would suffice to sample the black hole microstates in a controlled way. i.e. one could try to find a basis of geometries which are well-described and distinguishable in supergravity and which span the black hole microstates but for which each basis element is assigned a measure. In this approach, one would deal with the fact that many geometries are too similar to be distinguished in supergravity by picking representative geometries with appropriate measures. In constructing such a representative basis, the detailed matching between geometries and black hole microstates would be crucial, to correctly assign measures and to show that the basis indeed

spans all the black hole microstates.

The plan of this paper is as follows. In section 2 we determine the fuzzball geometries for D1-D5 on T^4 from dualizing type II F1-P solutions whilst in section 3 we obtain fuzzball geometries for D1-D5 on $K3$ from dualizing heterotic F1-P solutions. The resulting solutions are of the same form and are summarized in section 4; readers interested only in the solutions may skip sections 2 and 3. In section 5 we extract from the asymptotically AdS regions the dual field theory data, one point functions for chiral primaries. In section 6 we discuss the correspondence between geometries and R vacua, extending the proposal of [11, 12] and using the holographic vevs to test this proposal. In section 7 we discuss more generally the implications of our results for the fuzzball proposal. Finally there are a number of appendices. In appendix A we state our conventions for the field equations and duality rules, in appendix B we discuss in detail the reduction of type IIB on $K3$ and appendix C summarizes relevant properties of spherical harmonics. In appendix D we discuss fundamental string solutions with winding along the torus, and the corresponding duals in the D1-D5 system. In appendix E we derive the density of ground states with fixed R charges.

2. Fuzzball solutions on T^4

In this section we will obtain general 2-charge solutions for the D1-D5 system on T^4 from type II F1-P solutions.

2.1 Chiral null models

Let us begin with a general chiral null model of ten-dimensional supergravity, written in the form

$$\begin{aligned}
 ds^2 &= H^{-1}(x, v)dv(-du + K(x, v)dv + 2A_I(x, v)dx^I) + dx^I dx_I; \\
 e^{-2\Phi} &= H(x, v); \quad B_{uv}^{(2)} = \frac{1}{2}(H(x, v)^{-1} - 1); \quad B_{vI}^{(2)} = H(x, v)^{-1}A_I(x, v).
 \end{aligned}
 \tag{2.1}$$

The conventions for the supergravity field equations are given in the appendix A.1. The above is a solution of the equations of motion provided that the defining functions are harmonic in the transverse directions, labeled by x^I :

$$\square H(x, v) = \square K(x, v) = \square A_I(x, v) = (\partial_I A^I(x, v) - \partial_v H(x, v)) = 0.
 \tag{2.2}$$

Solutions of these equations appropriate for describing solitonic fundamental strings carrying momentum were given in [14, 15]:

$$H = 1 + \frac{Q}{|x - F(v)|^6}, \quad A_I = -\frac{Q\dot{F}_I(v)}{|x - F(v)|^6}, \quad K = \frac{Q^2\dot{F}(v)^2}{Q|x - F(v)|^6},
 \tag{2.3}$$

where $F^I(v)$ is an arbitrary null curve describing the transverse location of the string, and \dot{F}^I denotes $\partial_v F^I(v)$. More general solutions appropriate for describing solitonic strings with fermionic condensates were discussed in [13]. Here we will dualise without using the

explicit forms of the functions, thus the resulting dual supergravity solutions are applicable for all choices of harmonic functions.

The F1-P solutions described by such chiral null models can be dualised to give corresponding solutions for the D1-D5 system as follows. Compactify four of the transverse directions on a torus, such that x^i with $i = 1, \dots, 4$ are coordinates on R^4 and x^ρ with $\rho = 5, \dots, 8$ are coordinates on T^4 . Then let $v = (t - y)$ and $u = (t + y)$ with the coordinate y being periodic with length $L_y \equiv 2\pi R_y$, and smear all harmonic functions over both this circle and over the T^4 , so that they satisfy

$$\square_{R^4} H(x) = \square_{R^4} K(x) = \square_{R^4} A_I(x) = 0, \quad \partial_i A^i = 0. \quad (2.4)$$

Thus the harmonic functions appropriate for describing strings with only bosonic condensates are

$$\begin{aligned} H &= 1 + \frac{Q}{L_y} \int_0^{L_y} \frac{dv}{|x - F(v)|^2}; & A_i &= -\frac{Q}{L_y} \int_0^{L_y} \frac{dv \dot{F}_i(v)}{|x - F(v)|^2}; \\ A_\rho &= -\frac{Q}{L_y} \int_0^{L_y} \frac{dv \dot{F}_\rho(v)}{|x - F(v)|^2}; & K &= \frac{Q}{L_y} \int_0^{L_y} \frac{dv (\dot{F}_i(v)^2 + \dot{F}_\rho(v)^2)}{|x - F(v)|^2}. \end{aligned} \quad (2.5)$$

Here $|x - F(v)|^2$ denotes $\sum_i (x_i - F_i(v))^2$. Note that neither $\dot{F}_i(v)$ nor $\dot{F}_\rho(v)$ have zero modes; the asymptotic expansions of A_I at large $|x|$ therefore begin at order $1/|x|^3$. Closure of the curve in R^4 automatically implies that $\dot{F}_i(v)$ has no zero modes. The question of whether $\dot{F}_\rho(v)$ has zero modes is more subtle: since the torus coordinate x^ρ is periodic, the curve $F_\rho(v)$ could have winding modes. As we will discuss in appendix D, however, such winding modes are possible only when the worldsheet theory is deformed by constant B fields. The corresponding supergravity solutions, and those obtained from them by dualities, should thus not be included in describing BPS states in the original 2-charge systems.

The appropriate chain of dualities to the $D1 - D5$ system is

$$\begin{pmatrix} P_y \\ F1_y \end{pmatrix} \xrightarrow{S} \begin{pmatrix} P_y \\ D1_y \end{pmatrix} \xrightarrow{T_{5678}} \begin{pmatrix} P_y \\ D5_{y5678} \end{pmatrix} \xrightarrow{S} \begin{pmatrix} P_y \\ NS5_{y5678} \end{pmatrix} \xrightarrow{T_y} \begin{pmatrix} F1_y \\ NS5_{y5678} \end{pmatrix}, \quad (2.6)$$

to map to the type IIA NS5-F1 system. The subsequent dualities

$$\begin{pmatrix} F1_y \\ NS5_{y5678} \end{pmatrix} \xrightarrow{T_8} \begin{pmatrix} F1_y \\ NS5_{y5678} \end{pmatrix} \xrightarrow{S} \begin{pmatrix} D1_y \\ D5_{y5678} \end{pmatrix} \quad (2.7)$$

result in a D1-D5 system. Here the subscripts of $Dp_{a_1 \dots a_p}$ denote the spatial directions wrapped by the brane. In carrying out these dualities we use the rules reviewed in appendix A.2. We will give details of the intermediate solution in the type IIA NS5-F1 system since it differs from that obtained in [3].

2.2 The IIA F1-NS5 system

By dualizing the chiral null model from the F1-P system in IIB to F1-NS5 in IIA one

obtains the solution

$$\begin{aligned}
 ds^2 &= \tilde{K}^{-1}[-(dt - A_i dx^i)^2 + (dy - B_i dx^i)^2] + H dx_i dx^i + dx_\rho dx^\rho \\
 e^{2\Phi} &= \tilde{K}^{-1} H, & B_{ty}^{(2)} &= \tilde{K}^{-1} - 1, \\
 B_{\bar{\mu}i}^{(2)} &= \tilde{K}^{-1} \mathcal{B}_i^{\bar{\mu}}, & B_{ij}^{(2)} &= -c_{ij} + 2\tilde{K}^{-1} A_{[i} B_{j]} \\
 C_\rho^{(1)} &= H^{-1} A_\rho, & C_{ty\rho}^{(3)} &= (H\tilde{K})^{-1} A_\rho, & C_{\bar{\mu}i\rho}^{(3)} &= (H\tilde{K})^{-1} \mathcal{B}_i^{\bar{\mu}} A_\rho, \\
 C_{ij\rho}^{(3)} &= (\lambda_\rho)_{ij} + 2(H\tilde{K})^{-1} A_\rho A_{[i} B_{j]}, & C_{\rho\sigma\tau}^{(3)} &= \epsilon_{\rho\sigma\tau\pi} H^{-1} A^\pi,
 \end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
 \tilde{K} &= 1 + K - H^{-1} A_\rho A_\rho, & dc &= - *_4 dH, & dB &= - *_4 dA, \\
 d\lambda_\rho &= *_4 dA_\rho, & \mathcal{B}_i^{\bar{\mu}} &= (-B_i, A_i),
 \end{aligned} \tag{2.9}$$

with $\bar{\mu} = (t, y)$. Here the transverse and torus directions are denoted by (i, j) and (ρ, σ) respectively and $*_4$ denotes the Hodge dual in the flat metric on R^4 , with $\epsilon_{\rho\sigma\tau\pi}$ denoting the Hodge dual in flat T^4 metric. The defining functions satisfy the equations given in (2.4).

The RR field strengths corresponding to the above potentials are

$$\begin{aligned}
 F_{i\rho}^{(2)} &= \partial_i (H^{-1} A_\rho), & F_{tyi\rho}^{(4)} &= \tilde{K}^{-1} \partial_i (H^{-1} A_\rho), \\
 F_{\bar{\mu}ij\rho}^{(4)} &= 2\tilde{K}^{-1} \mathcal{B}_{[i}^{\bar{\mu}} \partial_{j]} (H^{-1} A_\rho), & F_{i\rho\sigma\tau}^{(4)} &= \epsilon_{\rho\sigma\tau\pi} \partial_i (H^{-1} A^\pi), \\
 F_{ijk\rho}^{(4)} &= \tilde{K}^{-1} \left(6A_{[i} B_j \partial_{k]} (H^{-1} A_\rho) + H \epsilon_{ijkl} \partial^l (H^{-1} A_\rho) \right).
 \end{aligned} \tag{2.10}$$

Thus the solution describes NS5-branes wrapping the y circle and the T^4 , bound to fundamental strings delocalized on the T^4 and wrapping the y circle, with additional excitations on the T^4 . These excitations break the T^4 symmetry by singling out a direction within the torus, and source multipole moments of the RR fluxes; the solution however has no net D-brane charges.

Now let us briefly comment on the relation between this solution and that presented in appendix B of [3].² The NS-NS sector fields agree, but the RR fields are different; in [3] they are given as 1, 3 and 5-form potentials. The relation of these potentials to field strengths (and the corresponding field equations) is not given in [3]. As reviewed in appendix A.2, in the presence of both electric and magnetic sources it is rather natural to use the so-called democratic formalisms of supergravity [16], in which one includes p -form field strengths with $p > 5$ along with constraints relating higher and lower form field strengths. Any solution written in the democratic formalism can be rewritten in terms of the standard formalism, appropriately eliminating the higher form field strengths. If one interprets the RR forms of [3] in this way, one does not however obtain a supergravity solution in the democratic formalism; the Hodge duality constraints between higher and lower form field strengths are not satisfied. Furthermore, one would not obtain from the RR fields of [3] the solution written here in the standard formalism, after eliminating the higher forms.

²We thank Samir Mathur for discussions on this issue.

2.3 Dualizing further to the D1-D5 system

The final steps in the duality chain are T-duality along a torus direction, followed by S-duality. When T-dualizing further along a torus direction to a F1-NS5 solution in IIB, the excitations along the torus mean that the dual solution depends explicitly on the chosen T-duality cycle in the torus. We will discuss the physical interpretation of the distinguished direction in section 4. In the following the T-duality is taken along the x^8 direction, resulting in the following D1-D5 system:

$$\begin{aligned}
 ds^2 &= \frac{f_1^{1/2}}{f_5^{1/2} \tilde{f}_1} [-(dt - A_i dx^i)^2 + (dy - B_i dx^i)^2] + f_1^{1/2} f_5^{1/2} dx_i dx^i + f_1^{1/2} f_5^{-1/2} dx_\rho dx^\rho \\
 e^{2\Phi} &= \frac{f_1^2}{f_5 \tilde{f}_1}, & B_{ty}^{(2)} &= \frac{\mathcal{A}}{f_5 \tilde{f}_1}, & B_{\bar{\mu}i}^{(2)} &= \frac{\mathcal{A} \mathcal{B}_{\bar{\mu}i}^{\bar{\mu}}}{f_5 \tilde{f}_1}, & (2.11) \\
 B_{ij}^{(2)} &= \lambda_{ij} + \frac{2\mathcal{A} A_{[i} B_{j]}}{f_5 \tilde{f}_1}, & B_{\alpha\beta}^{(2)} &= -\epsilon_{\alpha\beta\gamma} f_5^{-1} \mathcal{A}^\gamma, & B_{\alpha 8}^{(2)} &= f_5^{-1} \mathcal{A}_\alpha, \\
 C^{(0)} &= -f_1^{-1} \mathcal{A}, & C_{ty}^{(2)} &= 1 - \tilde{f}_1^{-1}, & C_{\bar{\mu}i}^{(2)} &= -\tilde{f}_1^{-1} \mathcal{B}_{\bar{\mu}i}^{\bar{\mu}}, \\
 C_{ij}^{(2)} &= c_{ij} - 2\tilde{f}_1^{-1} A_{[i} B_{j]}, & C_{tyij}^{(4)} &= \lambda_{ij} + \frac{\mathcal{A}}{f_5 \tilde{f}_1} (c_{ij} + 2A_{[i} B_{j]}), \\
 C_{\bar{\mu}ijk}^{(4)} &= \frac{3\mathcal{A}}{f_5 \tilde{f}_1} \mathcal{B}_{[i}^{\bar{\mu}} c_{jk]}, & C_{ty\alpha\beta}^{(4)} &= -\epsilon_{\alpha\beta\gamma} f_5^{-1} \mathcal{A}^\gamma, & C_{ty\alpha 8}^{(4)} &= f_5^{-1} \mathcal{A}_\alpha, \\
 C_{\alpha\beta\gamma 8}^{(4)} &= \epsilon_{\alpha\beta\gamma} f_5^{-1} \mathcal{A}, & C_{ij\alpha 8}^{(4)} &= (\lambda_\alpha)_{ij} + f_5^{-1} \mathcal{A}_\alpha c_{ij}, & C_{ij\alpha\beta}^{(4)} &= -\epsilon_{\alpha\beta\gamma} (\lambda^\gamma)_{ij} + f_5^{-1} \mathcal{A}^\gamma c_{ij},
 \end{aligned}$$

where

$$\begin{aligned}
 f_5 &\equiv H, & \tilde{f}_1 &= 1 + K - H^{-1}(\mathcal{A}_\alpha \mathcal{A}_\alpha + (\mathcal{A})^2), & f_1 &= \tilde{f}_1 + H^{-1}(\mathcal{A})^2, \\
 dc &= -*_4 dH, & dB &= -*_4 dA, & \mathcal{B}_{\bar{\mu}i}^{\bar{\mu}} &= (-B_i, A_i), & (2.12) \\
 d\lambda_\alpha &= *_4 d\mathcal{A}_\alpha, & d\lambda &= *_4 d\mathcal{A}.
 \end{aligned}$$

Here $\bar{\mu} = (t, y)$ and we denote A_8 as \mathcal{A} with the remaining A_ρ being denoted by \mathcal{A}_α where the index α runs over only 5, 6, 7. The Hodge dual over these coordinates is denoted by $\epsilon_{\alpha\beta\gamma}$. Explicit expressions for these defining harmonic functions in terms of variables of the D1-D5 system will be given in section 4.

The forms with components along the torus directions can be written more compactly as follows. Introduce a basis of self-dual and anti-self dual 2-forms on the torus such that

$$\omega^{\alpha\pm} = \frac{1}{\sqrt{2}} (dx^{4+\alpha\pm} \wedge dx^8 \pm *_4(dx^{4+\alpha\pm} \wedge dx^8)), \quad (2.13)$$

with $\alpha_\pm = 1, 2, 3$. These forms are normalized such that

$$\int_{T^4} \omega^{\alpha\pm} \wedge \omega^{\beta\pm} = \pm (2\pi)^4 V \delta^{\alpha\pm\beta\pm}, \quad (2.14)$$

where $(2\pi)^4 V$ is the volume of the torus. Then the potentials wrapping the torus directions can be expressed as

$$\begin{aligned} B_{\rho\sigma}^{(2)} &= C_{ty\rho\sigma}^{(4)} = \sqrt{2} f_5^{-1} \mathcal{A}^{\alpha-} \omega_{\rho\sigma}^{\alpha-}, \\ C_{ij\rho\sigma}^{(4)} &= \sqrt{2} ((\lambda_{ij})^{\alpha-} + f_5^{-1} \mathcal{A}^{\alpha-} c_{ij}) \omega_{\rho\sigma}^{\alpha-}, \\ C_{\rho\sigma\tau\pi}^{(4)} &= \epsilon_{\rho\sigma\tau\pi} f_5^{-1} \mathcal{A}, \end{aligned} \tag{2.15}$$

with $\epsilon_{\rho\sigma\tau\pi}$ being the Hodge dual in the flat metric on T^4 . Note that these fields are expanded only in the anti-self dual two-forms, with neither the self dual two-forms nor the odd-dimensional forms on the torus being switched on anywhere in the solution. As we will discuss later, this means the corresponding six-dimensional solution can be described in chiral $N = 4b$ six-dimensional supergravity. The components of forms associated with the odd cohomology of T^4 reduce to gauge fields in six dimensions which are contained in the full $N = 8$ six-dimensional supergravity, but not its truncation to $N = 4b$.

3. Fuzzball solutions on $K3$

In this section we will obtain general 2-charge solutions for the D1-D5 system on $K3$ from F1-P solutions of the heterotic string.

3.1 Heterotic chiral model in 10 dimensions

The chiral model for the charged heterotic F1-P system in 10 dimensions is:

$$\begin{aligned} ds^2 &= H^{-1}(-dudv + (K - 2\alpha' H^{-1} N^{(c)} N^{(c)})dv^2 + 2A_I dx^I dv) + dx_I dx^I \\ \hat{B}_{uv}^{(2)} &= \frac{1}{2}(H^{-1} - 1), \quad \hat{B}_{vI}^{(2)} = H^{-1} A_I, \\ \hat{\Phi} &= -\frac{1}{2} \ln H, \quad \hat{V}_v^{(c)} = H^{-1} N^{(c)}, \end{aligned} \tag{3.1}$$

where $I = 1, \dots, 8$ labels the transverse directions and $\hat{V}_m^{(c)}$ are Abelian gauge fields, with $((c) = 1, \dots, 16)$ labeling the elements of the Cartan of the gauge group. The fields are denoted with hats to distinguish them from the six-dimensional fields used in the next subsection. The equations of motion for the heterotic string are given in appendix A.1; here again the defining functions satisfy

$$\square H(x, v) = \square K(x, v) = \square A_I(x, v) = (\partial_I A^I(x, v) - \partial_v H(x, v)) = \square N^{(c)} = 0. \tag{3.2}$$

For the solution to correspond to a solitonic charged heterotic string, one takes the following solutions

$$\begin{aligned} H &= 1 + \frac{Q}{|x - F(v)|^6}, \quad A_I = -\frac{Q \dot{F}_I(v)}{|x - F(v)|^6}, \quad N^{(c)} = \frac{q^{(c)}(v)}{|x - F(v)|^6}, \\ K &= \frac{Q^2 \dot{F}(v)^2 + 2\alpha' q^{(c)} q^{(c)}(v)}{Q|x - F(v)|^6}, \end{aligned} \tag{3.3}$$

where $F^I(v)$ is an arbitrary null curve in R^8 ; $q^{(c)}(v)$ is an arbitrary charge wave and $\dot{F}_I(v)$ denotes $\partial_v F_I(v)$. Such solutions were first discussed in [14, 15], although the above has a more generic charge wave, lying in $\Upsilon(1)^{16}$ rather than $\Upsilon(1)$. In what follows it will be convenient to set $\alpha' = \frac{1}{4}$.

These solutions can be related by a duality chain to fuzzball solutions in the D1-D5 system compactified on $K3$. The chain of dualities is the following:

$$\begin{pmatrix} P_y \\ F1_y \end{pmatrix}_{Het, T^4} \rightarrow \begin{pmatrix} P_y \\ NS5_{ty, K3} \end{pmatrix}_{IIA} \xrightarrow{T_y} \begin{pmatrix} F1_y \\ NS5_{ty, K3} \end{pmatrix}_{IIB} \xrightarrow{S} \begin{pmatrix} D1_y \\ D5_{ty, K3} \end{pmatrix}_{IIB} \quad (3.4)$$

The first step in the duality is string-string duality between the heterotic theory on T^4 and type IIA on $K3$. Again the subscripts of $Dp_{a_1 \dots a_p}$ denote the spatial directions wrapped by the brane. To use this chain of dualities on the charged solitonic strings given above, the solutions must be smeared over the T^4 and over v , so that the harmonic functions satisfy

$$\square_{R^4} H = \square_{R^4} K = \square_{R^4} A_I = \square_{R^4} N^{(c)} = \partial_i A^i = 0 \quad (3.5)$$

where $i = 1, \dots, 4$ labels the transverse R^4 directions. Note that although the chain of dualities is shorter than in the previous case there are various subtleties associated with it, related to the $K3$ compactification, which will be discussed below.

3.2 Compactification on T^4

Compactification of the heterotic theory on T^4 is straightforward, see [18, 19] and the review [20]. The 10-dimensional metric is reduced as

$$\hat{G}_{mn} = \begin{pmatrix} g_{MN} + G_{\rho\sigma} V_M^{(1)\rho} V_N^{(1)\sigma} & V_M^{(1)\rho} G_{\rho\sigma} \\ V_N^{(1)\sigma} G_{\rho\sigma} & G_{\rho\sigma} \end{pmatrix}, \quad (3.6)$$

where $V_M^{(1)\rho}$ with $\rho = 1, \dots, 4$, are KK gauge fields. (Recall that the ten-dimensional quantities are denoted with hats to distinguish them from six-dimensional quantities.) The reduced theory contains the following bosonic fields: the graviton g_{MN} , the six-dimensional dilaton Φ_6 , 24 Abelian gauge fields $V_M^{(a)} \equiv (V_M^{(1)\rho}, V_{M\rho}^{(2)}, V_M^{(3)(c)})$, a two form B_{MN} and an $O(4, 20)$ matrix of scalars M . Note that the index $(a), (b)$ for the $SO(4, 20)$ vector runs from $(1, \dots, 24)$. These six-dimensional fields are related to the ten-dimensional fields as

$$\begin{aligned} \Phi_6 &= \hat{\Phi} - \frac{1}{2} \ln \det G_{\rho\sigma}; \\ V_{M\rho}^{(2)} &= \hat{B}_{M\rho}^{(2)} + \hat{B}_{\rho\sigma}^{(2)} V_M^{(1)\sigma} + \frac{1}{2} \hat{V}_\rho^{(c)} V_M^{(3)(c)}; \\ V_M^{(3)(c)} &= \hat{V}_M^{(c)} - \hat{V}_\rho^{(c)} V_M^{(1)\rho}; \\ H_{MNP} &= 3(\partial_{[M} \hat{B}_{NP]}^{(2)} - \frac{1}{2} V_{[M}^{(a)} L_{(a)(b)} F(V)_{NP]}^{(b)}), \end{aligned} \quad (3.7)$$

with the metric g_{MN} and $V_M^{(1)\rho}$ defined in (3.6). The matrix L is given by

$$L = \begin{pmatrix} I_4 & 0 \\ 0 & -I_{20} \end{pmatrix}, \quad (3.8)$$

where I_n denotes the $n \times n$ identity matrix. The scalar moduli are defined via

$$M = \Omega_1^T \begin{pmatrix} G^{-1} & -G^{-1}C & -G^{-1}V^T \\ -C^T G^{-1} & G + C^T G^{-1}C + V^T V & C^T G^{-1}V^T + V^T \\ -VG^{-1} & VG^{-1}C + V & I_{16} + VG^{-1}V^T \end{pmatrix} \Omega_1, \quad (3.9)$$

where $G \equiv [\hat{G}_{\rho\sigma}]$, $C \equiv [\frac{1}{2}\hat{V}_\rho^{(c)}\hat{V}_\sigma^{(c)} + \hat{B}_{\rho\sigma}^{(2)}]$ and $V \equiv [\hat{V}_\rho^{(c)}]$ are defined in terms of the components of the 10-dimensional fields along the torus. The constant $O(4, 20)$ matrix Ω_1 is given by

$$\Omega_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} I_4 & I_4 & 0 \\ -I_4 & I_4 & 0 \\ 0 & 0 & \sqrt{2}I_{16} \end{pmatrix}. \quad (3.10)$$

This matrix arises in (3.9) as follows. In [18, 20] the matrix L was chosen to be off-diagonal, but for our purposes it is useful for L to be diagonal. An off-diagonal choice is associated with an off-diagonal intersection matrix for the self-dual and anti-self-dual forms of $K3$, but this is an unnatural choice for our solutions, in which only anti-self-dual forms are active. Thus relative to the conventions of [18, 20] we take $L \rightarrow \Omega_1^T L \Omega_1$, which induces $M \rightarrow \Omega_1^T M \Omega_1$ and $F \rightarrow \Omega_1^T F$. The definitions of this and other constant matrices used throughout the paper are summarized in appendix B.2.

These fields satisfy the equations of motion following from the action

$$S = \frac{1}{2\kappa_6^2} \int d^6x \sqrt{-g} e^{-2\Phi_6} [R + 4(\partial\Phi_6)^2 - \frac{1}{12}H_3^2 - \frac{1}{4}F(V)_{MN}^{(a)}(LML)_{(a)(b)}F(V)^{(b)MN} + \frac{1}{8}\text{tr}(\partial_M ML \partial^M ML)], \quad (3.11)$$

where α' has been set to $1/4$ and $\kappa_6^2 = \kappa_{10}^2/V_4$ with V_4 the volume of the torus.

The reduction of the heterotic solution to six dimensions is then

$$\begin{aligned} ds^2 &= H^{-1} \left[-dudv + \left(K - H^{-1} \left(\frac{1}{2}(N^{(c)})^2 + (A_\rho)^2 \right) \right) dv^2 + 2A_i dx^i dv \right] + dx_i dx^i, \\ B_{uv} &= \frac{1}{2}(H^{-1} - 1), \quad B_{vi} = H^{-1}A_i, \quad \Phi_6 = -\frac{1}{2} \ln H \\ V_v^{(a)} &= \left(0_4, \sqrt{2}H^{-1}A_\rho, H^{-1}N^{(c)} \right), \quad M = I_{24}, \end{aligned} \quad (3.12)$$

where $i = 1, \dots, 4$ runs over the transverse R^4 directions and $\rho = 5, \dots, 8$ runs over the internal directions of the T^4 . Thus the six-dimensional solution has only one non-trivial scalar field, the dilaton, with all other scalar fields being constant.

3.3 String-string duality to P-NS5 (IIA) on $K3$

Given the six-dimensional heterotic solution, the corresponding IIA solution in six dimensions can be obtained as follows. Compactification of type IIA on $K3$ leads to the following six-dimensional theory [17]:

$$\begin{aligned} S' &= \frac{1}{2\kappa_6^2} \int d^6x \sqrt{-g'} \left(e^{-2\Phi'_6} \left[R' + 4(\partial\Phi'_6)^2 - \frac{1}{12}H_3'^2 + \frac{1}{8}\text{tr}(\partial_M M' L \partial^M M' L) \right] \right. \\ &\quad \left. - \frac{1}{4}F'(V)_{MN}^{(a)}(LM'L)_{(a)(b)}F'(V)^{(b)MN} \right) - 2 \int B'_2 \wedge F'_2(V)^{(a)} \wedge F'_2(V)^{(b)} L_{(a)(b)}. \end{aligned} \quad (3.13)$$

The field content is the same as for the heterotic theory in (3.11); note that in contrast to (3.7) there is no Chern-Simons term in the definition of the 3-form field strength, that is, $H'_{MNP} = 3\partial_{[M}B'_{NP]}$.

The rules for string-string duality are [17]:

$$\begin{aligned} \Phi'_6 &= -\Phi_6, & g'_{MN} &= e^{-2\Phi_6} g_{MN}, & M' &= M, & V_M'^{(a)} &= V_M^{(a)}, \\ H'_3 &= e^{-2\Phi_6} *_6 H_3; \end{aligned} \tag{3.14}$$

these transform the equations of motion derived from (3.11) into ones derived from the action (3.13).

Acting with this string-string duality on the heterotic solutions (3.12) yields, dropping the primes on IIA fields:

$$\begin{aligned} ds^2 &= -dudv + (K - H^{-1}((N^{(c)})^2/2 + (A_\rho)^2))dv^2 + 2A_i dx^i dv + H dx_i dx^i, \\ H_{vij} &= -\epsilon_{ijkl} \partial^k A^l, & H_{ijk} &= \epsilon_{ijkl} \partial^l H, & \Phi_6 &= \frac{1}{2} \ln H, \\ V_v^{(a)} &= (0_4, \sqrt{2}H^{-1}A_\rho, H^{-1}N^{(c)}), & M &= I_{24}, \end{aligned} \tag{3.15}$$

with ϵ_{ijkl} denoting the dual in the flat R^4 metric. This describes NS5-branes on type IIA, wrapped on K3 and on the circle direction y , carrying momentum along the circle direction.

3.4 T-duality to F1-NS5 (IIB) on K3

The next step in the duality chain is T-duality on the circle direction y to give an NS5-F1 solution of type IIB on $K3$. It is most convenient to carry out this step directly in six dimensions, using the results of [22] on T-duality of type II theories on $K3 \times S^1$.

Recall that type IIB compactified on $K3$ gives $d = 6$, $N = 4b$ supergravity coupled to 21 tensor multiplets, constructed by Romans in [23]. The bosonic field content of this theory is the graviton g_{MN} , 5 self-dual and 21 anti-self dual tensor fields and an $O(5,21)$ matrix of scalars \mathcal{M} which can be written in terms of a vielbein $\mathcal{M}^{-1} = V^T V$. Following the notation of [30] the bosonic field equations may be written as

$$\begin{aligned} R_{MN} &= 2P_M^{nr} P_N^{nr} + H_{MPQ}^n H_N^{PQ} + H_{MPQ}^r H_N^{PQ}, \\ \nabla^M P_M^{nr} &= Q^{Mnm} P_M^{mr} + Q^{Mrs} P_M^{ns} + \frac{\sqrt{2}}{3} H^{nMNP} H_{MNP}^r, \end{aligned} \tag{3.16}$$

along with Hodge duality conditions on the 3-forms

$$*_6 H_3^n = H_3^n, \quad *_6 H_3^r = -H_3^r, \tag{3.17}$$

In these equations (m, n) are $SO(5)$ vector indices running from 1 to 5 whilst (r, s) are $SO(21)$ vector indices running from 6 to 26. The 3-form field strengths are given by

$$H^n = G^A V_A^n; \quad H^r = G^A V_A^r, \tag{3.18}$$

where $A \equiv \{n, r\} = 1, \dots, 26$; $G^A = db^A$ are closed and the vielbein on the coset space $SO(5, 21)/(SO(5) \times SO(21))$ satisfies

$$V^T \eta V = \eta, \quad V = \begin{pmatrix} V_A^n \\ V_A^r \end{pmatrix}, \quad \eta = \begin{pmatrix} I_5 & 0 \\ 0 & -I_{21} \end{pmatrix}. \tag{3.19}$$

The associated connection is

$$dVV^{-1} = \begin{pmatrix} Q^{mn} & \sqrt{2}P^{ms} \\ \sqrt{2}P^{rn} & Q^{rs} \end{pmatrix}, \quad (3.20)$$

where Q^{mn} and Q^{rs} are antisymmetric and the off-diagonal block matrices P^{ms} and P^{rn} are transposed to each other. Note also that there is a freedom in choosing the vielbein; $\text{SO}(5) \times \text{SO}(21)$ transformations acting on H_3 and V as

$$V \rightarrow OV, \quad H_3 \rightarrow OH_3, \quad (3.21)$$

leave G_3 and \mathcal{M}^{-1} unchanged. Note that the field equations (3.16) can also be derived from the $\text{SO}(5, 21)$ invariant Einstein frame pseudo-action [21]

$$S = \frac{1}{2\kappa_6^2} \int d^6x \sqrt{-g} \left(R + \frac{1}{8} \text{tr}(\partial\mathcal{M}^{-1}\partial\mathcal{M}) - \frac{1}{3} G_{MNP}^A \mathcal{M}_{AB}^{-1} G^{BMNP} \right), \quad (3.22)$$

with the Hodge duality conditions (3.17) being imposed independently.

Now let us consider the T-duality relating a six-dimensional IIB solution to a six-dimensional IIA solution of (3.13); the corresponding rules were derived in [22]. Given that the six-dimensional IIA supergravity has only an $\text{SO}(4, 20)$ symmetry, relating IIB to IIA requires explicitly breaking the $\text{SO}(5, 21)$ symmetry of the IIB action down to $\text{SO}(4, 20)$. That is, one defines a conformal frame in which only an $\text{SO}(4, 20)$ subgroup is manifest and in which the action reads

$$S = \frac{1}{2\kappa_6^2} \int d^6x \sqrt{-g} \left\{ e^{-2\Phi} \left(R + 4(\partial\Phi)^2 + \frac{1}{8} \text{tr}(\partial M^{-1}\partial M) \right) + \frac{1}{2} \partial l^{(a)} M_{(a)(b)}^{-1} \partial l^{(b)} - \frac{1}{3} G_{MNP}^A \mathcal{M}_{AB}^{-1} G^{BMNP} \right\}. \quad (3.23)$$

The $\text{SO}(5, 21)$ matrix \mathcal{M}^{-1} has now been split up into the dilaton Φ , an $\text{SO}(4, 20)$ vector $l^{(a)}$ and an $\text{SO}(4, 20)$ matrix $M_{(a)(b)}^{-1}$, and we have chosen the parametrization

$$\mathcal{M}_{AB}^{-1} = \Omega_3^T \begin{pmatrix} e^{-2\Phi} + l^T M^{-1} l + \frac{1}{4} e^{2\Phi} l^4 & -\frac{1}{2} e^{2\Phi} l^2 & (l^T M^{-1})_{(b)} + \frac{1}{2} e^{2\Phi} l^2 (l^T L)_{(b)} \\ -\frac{1}{2} e^{2\Phi} l^2 & e^{2\Phi} & -e^{2\Phi} (l^T L)_{(b)} \\ (M^{-1} l)_{(a)} + \frac{1}{2} e^{2\Phi} l^2 (Ll)_{(a)} & -e^{2\Phi} (Ll)_{(a)} & M_{(a)(b)}^{-1} + e^{2\Phi} (Ll)_{(a)} (l^T L)_{(b)} \end{pmatrix} \Omega_3, \quad (3.24)$$

where $l^2 = l^{(a)} l^{(b)} L_{(a)(b)}$, $L_{(a)(b)}$ was defined in (3.8) and Ω_3 is a constant matrix defined in appendix B.2.

The fields Φ , $l^{(a)}$ and M^{-1} and half of the 3-forms can now be related to the IIA fields of section 3.3 by the following T-duality rules (given in terms of the 2-form potentials

b^A) [22]:

$$\begin{aligned}
 \tilde{g}_{yy} &= g_{yy}^{-1}, & \tilde{b}_{yM}^1 + \tilde{b}_{yM}^{26} &= \frac{1}{2}g_{yy}^{-1}g_{yM}, & (3.25) \\
 \tilde{g}_{yM} &= g_{yy}^{-1}B_{yM}, & \tilde{b}_{MN}^1 + \tilde{b}_{MN}^{26} &= \frac{1}{2}g_{yy}^{-1}(B_{MN} + 2(g_{y[M}B_{N]y})), \\
 \tilde{g}_{MN} &= g_{MN} - g_{yy}^{-1}(g_{yM}g_{yN} - B_{yM}B_{yN}), & \tilde{l}^{(a)} &= V_y^{(a)}, \\
 \tilde{\Phi} &= \Phi - \frac{1}{2}\log|g_{yy}|, & \tilde{M}_{(a)(b)}^{-1} &= M_{(a)(b)}^{-1}, \\
 \tilde{b}_{yM}^{(a)+1} &= \frac{1}{\sqrt{8}}(V_M^{(a)} - g_{yy}^{-1}V_y^{(a)}g_{yM}), & & (1 \leq (a) \leq 24),
 \end{aligned}$$

Here y is the T-duality circle, the six-dimensional index M excludes y and IIB fields are denoted by tildes to distinguish them from IIA fields. The other half of the tensor fields, that is $((\tilde{b}_{yM}^1 - \tilde{b}_{yM}^{26}), (\tilde{b}_{MN}^1 - \tilde{b}_{MN}^{26}), \tilde{b}_{MN}^{(a)+1}, \tilde{b}_{MN}^{(a)+1})$, can then be determined using the Hodge duality constraints (3.17).

We now have all the ingredients to obtain the T-dual of the IIA solution (3.15) along $y \equiv \frac{1}{2}(u - v)$. The IIA solution is expressed in terms of harmonic functions which also depend on the null coordinate v , and thus one needs to smear the solutions before dualizing. Note that it is the harmonic functions $(H, K, A^I, N^{(c)})$ which must be smeared over v , rather than the six-dimensional fields given in (3.15), since it is the former that satisfy linear equations and can therefore be superimposed.

The Einstein frame metric and three forms are given by

$$\begin{aligned}
 ds^2 &= \frac{1}{\sqrt{H\tilde{K}}}[-(dt - A_i dx^i)^2 + (dy - B_i dx^i)^2] + \sqrt{H\tilde{K}}dx_i dx^i, \\
 G_{tyi}^A &= \partial_i \left(\frac{n^A}{H\tilde{K}} \right), & G_{\bar{\mu}ij}^A &= -2\partial_{[i} \left(\frac{n^A}{H\tilde{K}} \mathcal{B}_{j]}^{\bar{\mu}} \right), & (3.26) \\
 G_{ijk}^A &= \epsilon_{ijkl} \partial^l n^A + 6\partial_{[i} \left(\frac{n^A}{H\tilde{K}} A_j B_{k]} \right),
 \end{aligned}$$

where

$$\begin{aligned}
 n^m &= \frac{1}{4}(H + K + 1, 0_4), & n^r &= \frac{1}{4}(-2A_\rho, -\sqrt{2}N^{(c)}, H - K - 1), & (3.27) \\
 \tilde{K} &= 1 + K - H^{-1} \left(\frac{1}{2}(N^{(c)})^2 + (A_\rho)^2 \right), & dB &= - *_4 dA, & \mathcal{B}_i^\mu &= (-B_i, A_i).
 \end{aligned}$$

Recall that $n = 1, \dots, 5$ and $r = 6, \dots, 26$ and $*_4$ denotes the dual on flat R^4 ; $\bar{\mu} = (t, y)$. The SO(4, 20) scalars are given by

$$\Phi = \frac{1}{2} \ln \frac{H}{\tilde{K}}, \quad l^{(a)} = \left(0_4, \sqrt{2}H^{-1}A_\rho, H^{-1}N^{(c)} \right), \quad M = I_{24}. \quad (3.28)$$

The SO(5, 21) scalar matrix $\mathcal{M}^{-1} = V^T V$ in (3.24) can then conveniently be expressed in terms of the vielbein

$$V = \Omega_3^T \left(\begin{array}{ccc} \sqrt{H^{-1}\tilde{K}} & 0 & 0 \\ -(\sqrt{H^3\tilde{K}})^{-1}(A_\rho^2 + \frac{1}{2}(N^{(c)})^2) & \sqrt{H\tilde{K}^{-1}} & -\sqrt{H\tilde{K}^{-1}}l^{(b)} \\ l^{(a)} & 0 & I_{24} \end{array} \right) \Omega_3. \quad (3.29)$$

3.5 S-duality to D1-D5 on $K3$

One further step in the duality chain is required to obtain the D1-D5 solution in type IIB, namely S duality. However, in the previous section the type II solutions have been given in six rather than ten dimensions. To carry out S duality one needs to specify the relationship between six and ten dimensional fields. Whilst the ten-dimensional $SL(2, R)$ symmetry is part of the six-dimensional symmetry group, its embedding into the full six-dimensional symmetry group is only defined once one specifies the uplift to ten dimensions. The details of the dimensional reduction are given in appendix B, with the six-dimensional S duality rules being given in (B.16); the S duality leaves the Einstein frame metric invariant, and acts as a constant rotation and similarity transformation on the three forms G^A and the matrix of scalars \mathcal{M} respectively. The S-dual solution is thus

$$\begin{aligned}
 ds^2 &= \frac{1}{\sqrt{f_5 \tilde{f}_1}} [-(dt - A_i dx^i)^2 + (dy - B_i dx^i)^2] + \sqrt{f_5 \tilde{f}_1} dx_i dx^i, & (3.30) \\
 G_{tyi}^A &= \partial_i \left(\frac{m^A}{f_5 \tilde{f}_1} \right), & G_{\tilde{\mu}ij}^A &= -2\partial_{[i} \left(\frac{m^A}{f_5 \tilde{f}_1} \mathcal{B}_{j]}^{\tilde{\mu}} \right), \\
 G_{ijk}^A &= \epsilon_{ijkl} \partial^l m^A + 6\partial_{[i} \left(\frac{m^A}{f_5 \tilde{f}_1} A_j B_{k]} \right),
 \end{aligned}$$

with

$$\begin{aligned}
 m^n &= \left(0_4, \frac{1}{4}(f_5 + F_1) \right), & (3.31) \\
 m^r &= \frac{1}{4} \left((f_5 - F_1), -2A_\alpha, -\sqrt{2}N^{(c)}, 2A_5 \right) \\
 &\equiv \frac{1}{4} \left((f_5 - F_1), -2\mathcal{A}^{\alpha-}, 2\mathcal{A} \right).
 \end{aligned}$$

Here the index $\alpha = 6, 7, 8$. Note that the specific reduction used here, see appendix B, distinguished A_5 from the other A_ρ and $N^{(c)}$. A different embedding would single out a different harmonic function, and hence a different vector, and it is thus convenient to introduce $(\mathcal{A}, \mathcal{A}^{\alpha-})$ to denote the choice of splitting more abstractly. Also as in (2.12) it is convenient to introduce the following combinations of harmonic functions:

$$\begin{aligned}
 f_5 &= H, & \tilde{f}_1 &= 1 + K - H^{-1}(\mathcal{A}^2 + \mathcal{A}^{\alpha-} \mathcal{A}^{\alpha-}), & (3.32) \\
 F_1 &= 1 + K, & f_1 &= \tilde{f}_1 + H^{-1} \mathcal{A}^2.
 \end{aligned}$$

The vielbein of scalars is given by

$$V = \Omega_4^T \begin{pmatrix} \sqrt{f_1^{-1} \tilde{f}_1} & 0 & 0 & 0 & 0 \\ GA^2 & \sqrt{\tilde{f}_1^{-1} f_1} & -GA F_1 & (\sqrt{f_1 \tilde{f}_1})^{-1} \mathcal{A} & -GA k^\gamma \\ -FA & 0 & \sqrt{f_5^{-1} f_1} & 0 & 0 \\ FA & 0 & -\frac{1}{2} f_5^{-1} F (k^\gamma)^2 & \sqrt{f_5 f_1^{-1}} & -F k^\gamma \\ 0 & 0 & f_5^{-1} k^\gamma & 0 & I_{22} \end{pmatrix} \Omega_4, & (3.33)$$

where to simplify notation quantities (F, G) are defined as

$$F = (f_1 f_5)^{-1/2}, \quad G = (f_1 \tilde{f}_1 f_5^2)^{-1/2}. \quad (3.34)$$

We also define the 22-dimensional vector k^γ as

$$k^\gamma = (0_3, \sqrt{2}\mathcal{A}^{\alpha-}). \quad (3.35)$$

Here $\gamma = 1, \dots, b^2$ where the second Betti number is $b^2 = 22$ for K3. Using the reduction formulae (B.13) and (B.14), the six-dimensional solution (3.30), (3.33) can be lifted to ten dimensions, resulting in a solution with an analogous form to the T^4 case (2.11). We will thus summarize the solution for both cases in the following section.

4. D1-D5 fuzzball solutions

In this section we will summarize the D1-D5 fuzzball solutions with internal excitations, for both the $K3$ and T^4 cases. In both cases the solutions can be written as

$$\begin{aligned} ds^2 &= \frac{f_1^{1/2}}{\tilde{f}_1 f_5^{1/2}} [-(dt - A_i dx^i)^2 + (dy - B_i dx^i)^2] + f_1^{1/2} f_5^{1/2} dx_i dx^i + f_1^{1/2} f_5^{-1/2} ds_{M^4}^2, \\ e^{2\Phi} &= \frac{f_1^2}{f_5 \tilde{f}_1}, & B_{ty}^{(2)} &= \frac{\mathcal{A}}{f_5 \tilde{f}_1}, & B_{\bar{\mu}i}^{(2)} &= \frac{\mathcal{A} \mathcal{B}_{\bar{\mu}i}^{\bar{\mu}}}{f_5 \tilde{f}_1}, & (4.1) \\ B_{ij}^{(2)} &= \lambda_{ij} + \frac{2\mathcal{A} A_{[i} B_{j]}}{f_5 \tilde{f}_1}, & B_{\rho\sigma}^{(2)} &= f_5^{-1} k^\gamma \omega_{\rho\sigma}^\gamma, & C^{(0)} &= -f_1^{-1} \mathcal{A}, \\ C_{ty}^{(2)} &= 1 - \tilde{f}_1^{-1}, & C_{\bar{\mu}i}^{(2)} &= -\tilde{f}_1^{-1} \mathcal{B}_{\bar{\mu}i}^{\bar{\mu}}, & C_{ij}^{(2)} &= c_{ij} - 2\tilde{f}_1^{-1} A_{[i} B_{j]}, \\ C_{tyij}^{(4)} &= \lambda_{ij} + \frac{\mathcal{A}}{f_5 \tilde{f}_1} (c_{ij} + 2A_{[i} B_{j]}), & C_{\bar{\mu}ijk}^{(4)} &= \frac{3\mathcal{A}}{f_5 \tilde{f}_1} \mathcal{B}_{\bar{\mu}[i}^{\bar{\mu}} c_{jk]}, \\ C_{ty\rho\sigma}^{(4)} &= f_5^{-1} k^\gamma \omega_{\rho\sigma}^\gamma, & C_{ij\rho\sigma}^{(4)} &= (\lambda_{ij} + f_5^{-1} k^\gamma c_{ij}) \omega_{\rho\sigma}^\gamma, & C_{\rho\sigma\tau\pi}^{(4)} &= f_5^{-1} \mathcal{A} \epsilon_{\rho\sigma\tau\pi}, \end{aligned}$$

where we introduce a basis of self-dual and anti-self-dual 2-forms $\omega^\gamma \equiv (\omega^{\alpha+}, \omega^{\alpha-})$ with $\gamma = 1, \dots, b^2$ on the compact manifold M^4 . For both T^4 and $K3$ the self-dual forms are labeled by $\alpha_+ = 1, 2, 3$ whilst the anti-self-dual forms are labeled by $\alpha_- = 1, 2, 3$ for T^4 and $\alpha_- = 1, \dots, 19$ for $K3$. The intersections and normalizations of these forms are defined in (2.13), (2.14) and (B.4). The solutions are expressed in terms of the following combinations of harmonic functions $(H, K, A_i, \mathcal{A}, \mathcal{A}^{\alpha-})$

$$\begin{aligned} f_5 &= H; & \tilde{f}_1 &= 1 + K - H^{-1}(\mathcal{A}^2 + \mathcal{A}^{\alpha-} \mathcal{A}^{\alpha-}); & f_1 &= \tilde{f}_1 + H^{-1} \mathcal{A}^2; \\ k^\gamma &= (0_3, \sqrt{2}\mathcal{A}^{\alpha-}); & dB &= - *_4 d\mathcal{A}; & dc &= - *_4 df_5; & (4.2) \\ d\lambda^\gamma &= *_4 dk^\gamma; & d\lambda &= *_4 d\mathcal{A}; & \mathcal{B}_{\bar{\mu}i}^{\bar{\mu}} &= (-B_i, A_i), \end{aligned}$$

where $\bar{\mu} = (t, y)$ and the Hodge dual $*_4$ is defined over (flat) R^4 , with the Hodge dual in the Ricci flat metric on the compact manifold being denoted by $\epsilon_{\rho\sigma\tau\pi}$. The constant term in $C_{ty}^{(2)}$ is chosen so that the potential vanishes at asymptotically flat infinity. The

corresponding RR field strengths are

$$\begin{aligned}
F_i^{(1)} &= -\partial_i (f_1^{-1} \mathcal{A}), & F_{tyi}^{(3)} &= (f_1 \tilde{f}_1 f_5^2)^{-1} \left(f_5^2 \partial_i \tilde{f}_1 + f_5 \mathcal{A} \partial_i \mathcal{A} - \mathcal{A}^2 \partial_i f_5 \right), \\
F_{\tilde{\mu}ij}^{(3)} &= (f_5^2 f_1 \tilde{f}_1)^{-1} \left(2\mathcal{B}_{[i}^{\tilde{\mu}} (f_5 \partial_{j]} \tilde{f}_1 + f_5 \mathcal{A} \partial_{j]} \mathcal{A} - \mathcal{A}^2 \partial_{j]} f_5 \right) + 2\tilde{f}_1 f_5^2 \partial_{[i} \mathcal{B}_{j]}^{\tilde{\mu}}, \\
F_{ijk}^{(3)} &= -\epsilon_{ijkl} (\partial^l f_5 - f_1^{-1} \mathcal{A} \partial^l \mathcal{A}) - 6f_1^{-1} \partial_{[i} (A_j B_{k]}) \\
&\quad + (f_5^2 f_1 \tilde{f}_1)^{-1} \left(6A_{[i} B_j (f_5 \partial_{k]} \tilde{f}_1 + f_5 \mathcal{A} \partial_{k]} \mathcal{A} - \mathcal{A}^2 \partial_{k]} f_5 \right), \\
F_{i\rho\sigma}^{(3)} &= f_1^{-1} \mathcal{A} \partial_i (f_5^{-1} k^\gamma) \omega_{\rho\sigma}^\gamma, \\
F_{i\rho\sigma\tau\pi}^{(5)} &= \epsilon_{\rho\sigma\tau\pi} \partial_i (f_5^{-1} \mathcal{A}), & F_{tyijk}^{(5)} &= \epsilon_{ijkl} \tilde{f}_1^{-1} f_5 \partial^l (f_5^{-1} \mathcal{A}), \\
F_{\tilde{\mu}ijkl}^{(5)} &= -\epsilon_{ijkl} f_5 \tilde{f}_1^{-1} \mathcal{B}_m^{\tilde{\mu}} \partial^m (f_5^{-1} \mathcal{A}), \\
F_{tyi\rho\sigma}^{(5)} &= \tilde{f}_1^{-1} \partial_i (k^\gamma / f_5) \omega_{\rho\sigma}^\gamma, & F_{\tilde{\mu}ij\rho\sigma}^{(5)} &= 2\tilde{f}_1^{-1} \mathcal{B}_{[i}^{\tilde{\mu}} \partial_{j]} (f_5^{-1} k^\gamma) \omega_{\rho\sigma}^\gamma, \\
F_{ijk\rho\sigma}^{(5)} &= \left(6\tilde{f}_1^{-1} A_{[i} B_j \partial_{k]} (f_5^{-1} k^\gamma) + \epsilon_{ijkl} f_5 \partial^l (f_5^{-1} k^\gamma) \right) \omega_{\rho\sigma}^\gamma.
\end{aligned} \tag{4.3}$$

It has been explicitly checked that this is a solution of the ten-dimensional field equations for any choices of harmonic functions $(H, K, A_i, \mathcal{A}, \mathcal{A}^{\alpha-})$ with $\partial_i A^i = 0$. Note that in the case of $K3$ one needs the identity (B.15) for the harmonic forms to check the components of the Einstein equation along $K3$.

We are interested in solutions for which the defining harmonic functions are given by

$$\begin{aligned}
H &= 1 + \frac{Q_5}{L} \int_0^L \frac{dv}{|x - F(v)|^2}; & A_i &= -\frac{Q_5}{L} \int_0^L \frac{dv \dot{F}_i(v)}{|x - F(v)|^2}, \\
\mathcal{A} &= -\frac{Q_5}{L} \int_0^L \frac{dv \dot{\mathcal{F}}(v)}{|x - F(v)|^2}; & \mathcal{A}^{\alpha-} &= -\frac{Q_5}{L} \int_0^L \frac{dv \dot{\mathcal{F}}^{\alpha-}(v)}{|x - F(v)|^2}, \\
K &= \frac{Q_5}{L} \int_0^L \frac{dv (\dot{F}(v)^2 + \dot{\mathcal{F}}(v)^2 + \dot{\mathcal{F}}^{\alpha-}(v)^2)}{|x - F(v)|^2}.
\end{aligned} \tag{4.4}$$

In these expressions Q_5 is the 5-brane charge and L is the length of the defining curve in the D1-D5 system, given by

$$L = 2\pi Q_5 / R, \tag{4.5}$$

where R is the radius of the y circle. Note that Q_5 has dimensions of length squared and is related to the integral charge via

$$Q_5 = \alpha' n_5 \tag{4.6}$$

(where g_s has been set to one). Assuming that the curves $(\dot{\mathcal{F}}(v), \dot{\mathcal{F}}^{\alpha-}(v))$ do not have zero modes, the D1-brane charge Q_1 is given by

$$Q_1 = \frac{Q_5}{L} \int_0^L dv (\dot{F}(v)^2 + \dot{\mathcal{F}}(v)^2 + \dot{\mathcal{F}}^{\alpha-}(v)^2), \tag{4.7}$$

and the corresponding integral charge is given by

$$Q_1 = \frac{n_1 (\alpha')^3}{V}, \tag{4.8}$$

where $(2\pi)^4 V$ is the volume of the compact manifold. The mapping of the parameters from the original F1-P systems to the D1-D5 systems was discussed in [1] and is unchanged here. The fact that the solutions take exactly the same form, regardless of whether the compact manifold is T^4 or $K3$, is unsurprising given that only zero modes of the compact manifold are excited.

The solutions defined in terms of the harmonic functions (4.4) describe the complete set of two-charge fuzzballs for the D1-D5 system on $K3$. In the case of T^4 , these describe fuzzballs with only bosonic excitations; the most general solution would include fermionic excitations and thus more general harmonic functions of the type discussed in [13]. Solutions involving harmonic functions with disconnected sources would be appropriate for describing Coulomb branch physics. Note that, whilst the solutions obtained by dualities from supersymmetric F1-P solutions are guaranteed to be supersymmetric, one would need to check supersymmetry explicitly for solutions involving other choices of harmonic functions.

In the final solutions one of the harmonic functions \mathcal{A} describing internal excitations is singled out from the others. In the original F1-P system, the solutions pick out a direction in the internal space. For the type II system on T^4 , the choice of A_ρ singles out a direction in the torus whilst in the heterotic solution the choice of $(A_\rho, N^{(c)})$ singles out a direction in the 20d internal space. Both duality chains, however, also distinguish directions in the internal space. In the T^4 case one had to choose a direction in the torus, whilst in the $K3$ case the choice is implicitly made when one uplifts type IIB solutions from six to ten dimensions. In particular, the uplift splits the 21 anti-self-dual six-dimensional 3-forms into $19 + 1 + 1$ associated with the ten-dimensional $(F^{(5)}, F^{(3)}, H^{(3)})$ respectively.

When there are no internal excitations, the final solutions must be independent of the choice of direction made in the duality chains but this does not remain true when the original solution breaks the rotational symmetry in the internal space. \mathcal{A} is the component of the original vector along the direction distinguished in the duality chain, whilst \mathcal{A}^α are the components orthogonal to this direction. When there are no excitations along the direction picked out by the duality, i.e. $\mathcal{A} = 0$, the solution considerably simplifies, becoming

$$\begin{aligned}
 ds^2 &= \frac{1}{(f_1 f_5)^{1/2}} [-(dt - A_i dx^i)^2 + (dy - B_i dx^i)^2] + f_1^{1/2} f_5^{1/2} dx_i dx^i + f_1^{1/2} f_5^{-1/2} ds_{M^4}^2, \\
 e^{2\Phi} &= \frac{f_1}{f_5}, \quad B_{\rho\sigma}^{(2)} = f_5^{-1} k^\gamma \omega_{\rho\sigma}^\gamma, \quad C_{ty}^{(2)} = 1 - f_1^{-1}, \quad C_{\bar{\mu}i}^{(2)} = -f_1^{-1} \mathcal{B}_i^{\bar{\mu}}, \\
 C_{ij}^{(2)} &= c_{ij} - 2f_1^{-1} A_{[i} B_{j]}, \quad C_{ty\rho\sigma}^{(4)} = f_5^{-1} k^\gamma \omega_{\rho\sigma}^\gamma, \quad C_{ij\rho\sigma}^{(4)} = (\lambda_{ij}^\gamma + f_5^{-1} k^\gamma c_{ij}) \omega_{\rho\sigma}^\gamma.
 \end{aligned}$$

In this solution the internal excitations induce fluxes of the NS 3-form and RR 5-form along anti-self dual cycles in the compact manifold (but no net 3-form or 5-form charges). By contrast the excitations parallel to the duality direction induce a field strength for the RR axion, NS 3-form field strength in the non-compact directions and RR 5-form field strength along the compact manifold (but again no net charges).

Let us also comment on the M^4 moduli in our solutions. The solutions are expressed in terms of a Ricci flat metric on M^4 and anti-self dual harmonic two forms. The forms

satisfy

$$\omega_{\rho\sigma}^{\gamma}\omega^{\delta\rho\sigma} = D^{\epsilon}_{\delta}d_{\gamma\epsilon} \equiv \delta_{\gamma\delta}, \quad (4.9)$$

where the intersection matrix $d_{\delta\gamma}$ and the matrix D^{γ}_{δ} relating the basis of forms and dual forms are defined in (B.4) and (B.6) respectively. The latter condition on D^{γ}_{ϵ} arose from the duality chain, and followed from the fact that in the original F1-P solutions the internal manifold had a flat square metric. Thus, the final solutions are expressed at a specific point in the moduli space of M^4 because the original F1-P solutions have specific fixed moduli. It is straightforward to extend the solutions to general moduli: one needs to change

$$\tilde{f}_1 = 1 + K - H^{-1}(\mathcal{A}^2 + \mathcal{A}^{\alpha-}\mathcal{A}^{\alpha-}) \rightarrow 1 + K - H^{-1}\left(\mathcal{A}^2 + \frac{1}{2}k^{\gamma}k^{\delta}D^{\epsilon}_{\delta}d_{\gamma\epsilon}\right), \quad (4.10)$$

with k^{γ} as defined in (4), to obtain the solution for more general D^{γ}_{δ} .

Given a generic fuzzball solution, one would like to check whether the geometry is indeed smooth and horizon-free. For the fuzzballs with no internal excitations this question was discussed in [3], the conclusion being that the solutions are non-singular unless the defining curve $F^i(v)$ is non-generic and self-intersects. In the appendix of [3], the smoothness of fuzzballs with internal excitations was also discussed. However, their D1-D5 solutions were incomplete: only the metric was given, and this was effectively given in the form (3.30) rather than (4.1). Nonetheless, their conclusion remains unchanged: following the same discussion as in [3] one can show that a generic fuzzball solution with internal excitations is non-singular provided that the defining curve $F^i(v)$ does not self-intersect and $\dot{F}_i(v)$ only has isolated zeroes. In particular, if there are no transverse excitations, $F^i(v) = 0$, the solution will be singular as discussed in section 6.6.

One can show that there are no horizons as follows. The harmonic function f_5 is clearly positive definite, by its definition. The functions (f_1, \tilde{f}_1) are also positive definite, since they can be rewritten as a sum of positive terms as

$$\begin{aligned} f_5\tilde{f}_1 &= \left(1 + \frac{Q_5}{L} \int_0^L \frac{dv}{|x-F|^2}\right) \left(1 + \frac{Q_5}{L} \int_0^L \frac{dv\dot{F}^2}{|x-F|^2}\right) \\ &+ \frac{Q_5}{L} \int_0^L \frac{dv(\dot{\mathcal{F}}(v))^2 + (\dot{\mathcal{F}}^{\alpha-}(v))^2}{|x-F|^2} \\ &+ \frac{1}{2}\left(\frac{Q_5}{L}\right)^2 \int_0^L \int_0^L \frac{dvdv'(\dot{\mathcal{F}}(v) - \dot{\mathcal{F}}(v'))^2 + (\dot{\mathcal{F}}^{\alpha-}(v) - \dot{\mathcal{F}}^{\alpha-}(v'))^2}{|x-F(v)|^2|x-F(v')|^2}, \end{aligned} \quad (4.11)$$

and a corresponding expression for f_5f_1 . Note that in the decoupling limit only the terms proportional to Q_5^2 remain, and these are also manifestly positive definite. Given that the defining functions have no zeroes anywhere, the geometry therefore has no horizons.

Now let us consider the conserved charges. From the asymptotics one can see that the fuzzball solutions have the same mass and D1-brane, D5-brane charges as the naive solution; the latter are given in (4.6) and (4.8) whilst the ADM mass is

$$M = \frac{\Omega_3 L_y}{\kappa_6^2}(Q_1 + Q_5), \quad (4.12)$$

where $L_y = 2\pi R$, $\Omega_3 = 2\pi^2$ is the volume of a unit 3-sphere, and $2\kappa_6^2 = (2\kappa^2)/(V(2\pi)^4)$ with $2\kappa^2 = (2\pi)^7(\alpha')^4$ in our conventions. The fuzzball solutions have in addition angular momenta, given by

$$J^{ij} = \frac{\Omega_3 L_y}{\kappa_6^2 L} \int_0^L dv (F^i \dot{F}^j - F^j \dot{F}^i). \tag{4.13}$$

These are the only charges; the fields $F^{(1)}$ and $F^{(5)}$ fall off too quickly at infinity for the corresponding charges to be non-zero. One can compute from the harmonic expansions of the fields dipole and more generally multipole moments of the charge distributions. A generic solution breaks completely the $SO(4)$ rotational invariance in R^4 , and this symmetry breaking is captured by these multipole moments.

However, the multipole moments computed at asymptotically flat infinity do not have a direct interpretation in the dual field theory. In contrast, the asymptotics of the solutions in the decoupling limit do give field theory information: one-point functions of chiral primaries are expressed in terms of the asymptotic expansions (and hence multipole moments) near the $AdS_3 \times S^3$ boundary. Thus it is more useful to compute in detail the latter, as we shall do in the next section.

5. Vevs for the fuzzball solutions

In the decoupling limit all of the fuzzball solutions are asymptotic to $AdS_3 \times S^3 \times M^4$, where M^4 is T^4 or $K3$. Therefore one can use AdS/CFT methods to extract holographic data from the geometries. In particular, the asymptotics of the six-dimensional solutions near the $AdS_3 \times S^3$ boundary encode the vevs of chiral primary operators in the dual field theory.

The precise relationship between asymptotics and vevs is however rather subtle. A systematic method for extracting vevs from asymptotically $AdS \times X$ solutions (with X an arbitrary compact manifold) was only recently constructed, in [10], building on earlier work [24–28], see also the review [29]. This method of Kaluza-Klein holography was then applied to the case of asymptotically $AdS_3 \times S^3$ solutions of $d = 6, N = 4b$ supergravity coupled to n_t tensor multiplets in [11, 12] and in what follows we will make use of many of the results derived there.

For fuzzball solutions on $K3$, the relevant solution of six-dimensional $N = 4b$ supergravity coupled to 21 tensor multiplets was given explicitly in (3.30). For the case of T^4 , we obtained the solution in ten dimensions, but there is a corresponding six-dimensional solution of $N = 4b$ supergravity coupled to 5 tensor multiplets. This solution is of exactly the same form as the $K3$ solution given in (3.30), but with the index $\alpha_- = 1, 2, 3$. Thus in what follows we will analyze both cases simultaneously. As mentioned earlier, the T^4 solution reduces to a solution of $d = 6, N = 4b$ supergravity rather than a solution of $d = 6, N = 8$ supergravity because forms associated with the odd cohomology of T^4 (and hence six-dimensional vectors) are not present in our solutions.

5.1 Holographic relations for vevs

Consider an $AdS_3 \times S^3$ solution of the six-dimensional field equations (3.16), such that

$$\begin{aligned} ds_6^2 &= \sqrt{Q_1 Q_5} \left(\frac{1}{z^2} (-dt^2 + dy^2 + dz^2) + d\Omega_3^2 \right); \\ G^5 &= H^5 \equiv g^{o5} = \sqrt{Q_1 Q_5} (r dr \wedge dt \wedge dy + d\Omega_3), \end{aligned} \quad (5.1)$$

with the vielbein being diagonal and all other three forms (both self-dual and anti-self dual) vanishing. In what follows it is convenient to absorb the curvature radius $\sqrt{Q_1 Q_5}$ into an overall prefactor in the action, and work with the unit radius $AdS_3 \times S^3$. Now express the perturbations of the six-dimensional supergravity fields relative to the $AdS_3 \times S^3$ background as

$$\begin{aligned} g_{MN} &= g_{MN}^o + h_{MN}; & G^A &= g^{oA} + g^A; \\ V_A^n &= \delta_A^n + \phi^{nr} \delta_A^r + \frac{1}{2} \phi^{nr} \phi^{mr} \delta_A^m; \\ V_A^r &= \delta_A^r + \phi^{nr} \delta_A^n + \frac{1}{2} \phi^{nr} \phi^{ns} \delta_A^s. \end{aligned} \quad (5.2)$$

These fluctuations can then be expanded in spherical harmonics as follows:

$$\begin{aligned} h_{\mu\nu} &= \sum h_{\mu\nu}^I(x) Y^I(y), \\ h_{\mu a} &= \sum (h_{\mu}^{Iv}(x) Y_a^{Iv}(y) + h_{(s)\mu}^I(x) D_a Y^I(y)), \\ h_{(ab)} &= \sum (\rho^{It}(x) Y_{(ab)}^{It}(y) + \rho_{(v)}^{Iv}(x) D_a Y_b^{Iv}(y) + \rho_{(s)}^I(x) D_{(a} D_{b)} Y^I(y)), \\ h_a^a &= \sum \pi^I(x) Y^I(y), \\ g_{\mu\nu\rho}^A &= \sum 3D_{[\mu} b_{\nu\rho]}^{(A)I}(x) Y^I(y), \\ g_{\mu\nu a}^A &= \sum (b_{\mu\nu}^{(A)I}(x) D_a Y^I(y) + 2D_{[\mu} Z_{\nu]}^{(A)Iv}(x) Y_a^{Iv}(y)); \\ g_{\mu ab}^A &= \sum (D_{\mu} U^{(A)I}(x) \epsilon_{abc} D^c Y^I(y) + 2Z_{\mu}^{(A)Iv} D_{[b} Y_{a]}^{Iv}); \\ g_{abc}^A &= \sum (-\epsilon_{abc} \Lambda^I U^{(A)I}(x) Y^I(y)); \\ \phi^{mr} &= \sum \phi^{(mr)I}(x) Y^I(y), \end{aligned} \quad (5.3)$$

Here (μ, ν) are AdS indices and (a, b) are S^3 indices, with x denoting AdS coordinates and y denoting sphere coordinates. The subscript (ab) denotes symmetrization of indices a and b with the trace removed. Relevant properties of the spherical harmonics are reviewed in appendix C. We will often use a notation where we replace the index I by the degree of the harmonic k or by a pair of indices (k, I) where k is the degree of the harmonic and I now parametrizes their degeneracy, and similarly for I_v, I_t .

Imposing the de Donder gauge condition $D^A h_{aM} = 0$ on the metric fluctuations removes the fields with subscripts (s, v) . In deriving the spectrum and computing correlation functions, this is therefore a convenient choice. The de Donder gauge choice is however not always a convenient choice for the asymptotic expansion of solutions; indeed the natural coordinate choice in our application takes us outside de Donder gauge. As discussed in [10]

this issue is straightforwardly dealt with by working with gauge invariant combinations of the fluctuations.

Next let us briefly review the linearized spectrum derived in [30], focusing on fields dual to chiral primaries. Consider first the scalars. It is useful to introduce the following combinations which diagonalize the linearized equations of motion:

$$s_I^{(r)k} = \frac{1}{4(k+1)}(\phi_I^{(5r)k} + 2(k+2)U_I^{(r)k}), \tag{5.4}$$

$$\sigma_I^k = \frac{1}{12(k+1)}(6(k+2)\hat{U}_I^{(5)k} - \hat{\pi}_I^k),$$

The fields $s^{(r)k}$ and σ^k correspond to scalar chiral primaries, with the masses of the scalar fields being

$$m_{s^{(r)k}}^2 = m_{\sigma^k}^2 = k(k-2), \tag{5.5}$$

The index r spans $6 \cdots 5 + n_t$ with $n_t = 5, 21$ respectively for T^4 and $K3$. Note also that $k \geq 1$ for $s^{(r)k}$; $k \geq 2$ for σ^k . The hats ($\hat{U}_I^{(5)k}, \hat{\pi}_I^k$) denote the following. As discussed in [10], the equations of motion for the gauge invariant fields are precisely the same as those in de Donder gauge, provided one replaces all fields with the corresponding gauge invariant field. The hat thus denotes the appropriate gauge invariant field, which reduces to the de Donder gauge field when one sets to zero all fields with subscripts (s, v) . For our purposes we will need these gauge invariant quantities only to leading order in the fluctuations, with the appropriate combinations being

$$\hat{\pi}_2^I = \pi_2^I + \Lambda^2 \rho_{2(s)}^I; \tag{5.6}$$

$$\hat{U}_2^{(5)I} = U_2^{(5)I} - \frac{1}{2} \rho_{2(s)}^I;$$

$$\hat{h}_{\mu\nu}^0 = h_{\mu\nu}^0 - \sum_{\alpha, \pm} h_\mu^{1\pm\alpha} h_\nu^{1\pm\alpha}.$$

Next consider the vector fields. It is useful to introduce the following combinations which diagonalize the equations of motion:

$$h_{\mu I_v}^\pm = \frac{1}{2}(C_{\mu I_v}^\pm - A_{\mu I_v}^\pm), \quad Z_{\mu I_v}^{(5)\pm} = \pm \frac{1}{4}(C_{\mu I_v}^\pm + A_{\mu I_v}^\pm). \tag{5.7}$$

For general k the equations of motion are Proca-Chern-Simons equations which couple (A_μ^\pm, C_μ^\pm) via a first order constraint [30]. The three dynamical fields at each degree k have masses $(k-1, k+1, k+3)$, corresponding to dual operators of dimensions $(k, k+2, k+4)$ respectively; the operators of dimension k are vector chiral primaries. The lowest dimension operators are the R symmetry currents, which couple to the $k=1$ $A_\mu^{\pm\alpha}$ bulk fields. The latter satisfy the Chern-Simons equation

$$F_{\mu\nu}(A^{\pm\alpha}) = 0, \tag{5.8}$$

where $F_{\mu\nu}(A^{\pm\alpha})$ is the curvature of the connection and the index $\alpha = 1, 2, 3$ is an SU(2) adjoint index. We will here only discuss the vevs of these vector chiral primaries.

Finally there is a tower of KK gravitons with $m^2 = k(k + 2)$ but only the massless graviton, dual to the stress energy tensor, will play a role here. Note that it is the combination $\hat{H}_{\mu\nu} = \hat{h}_{\mu\nu}^0 + \pi^0 g_{\mu\nu}^0$ which satisfies the Einstein equation; moreover one needs the appropriate gauge covariant combination $\hat{h}_{\mu\nu}^0$ given in (5.6).

Let us denote by $(\mathcal{O}_{S_I^{(r)k}}, \mathcal{O}_{\Sigma_I^k})$ the chiral primary operators dual to the fields $(s_I^{(r)k}, \sigma_I^k)$ respectively. The vevs of the scalar operators with dimension two or less can then be expressed in terms of the coefficients in the asymptotic expansion as

$$\begin{aligned} \langle \mathcal{O}_{S_i^{(r)1}} \rangle &= \frac{2N}{\pi} \sqrt{2} [s_i^{(r)1}]_1; & \langle \mathcal{O}_{S_I^{(r)2}} \rangle &= \frac{2N}{\pi} \sqrt{6} [s_I^{(r)2}]_2; \\ \langle \mathcal{O}_{\Sigma_I^2} \rangle &= \frac{N}{\pi} \left(2\sqrt{2} [\sigma_I^2]_2 - \frac{1}{3} \sqrt{2} a_{Iij} \sum_r [s_i^{(r)1}]_1 [s_j^{(r)1}]_1 \right). \end{aligned} \tag{5.9}$$

Here $[\psi]_n$ denotes the coefficient of the z^n term in the asymptotic expansion of the field ψ . The coefficient a_{Iij} refers to the triple overlap between spherical harmonics, defined in (C.5). Note that dimension one scalar spherical harmonics have degeneracy four, and are thus labeled by $i = 1, \dots, 4$.

Now consider the stress energy tensor and the R symmetry currents. The three dimensional metric and the Chern-Simons gauge fields admit the following asymptotic expansions

$$\begin{aligned} ds_3^2 &= \frac{dz^2}{z^2} + \frac{1}{z^2} \left(g_{(0)\bar{\mu}\bar{\nu}} + z^2 \left(g_{(2)\bar{\mu}\bar{\nu}} + \log(z^2) h_{(2)\bar{\mu}\bar{\nu}} + (\log(z^2))^2 \tilde{h}_{(2)\bar{\mu}\bar{\nu}} \right) + \dots \right) dx^{\bar{\mu}} dx^{\bar{\nu}}; \\ A^{\pm\alpha} &= \mathcal{A}^{\pm\alpha} + z^2 A_{(2)}^{\pm\alpha} + \dots \end{aligned} \tag{5.10}$$

The vevs of the R symmetry currents $J_u^{\pm\alpha}$ are then given in terms of terms in the asymptotic expansion of $A_\mu^{\pm\alpha}$ as

$$\langle J_{\bar{\mu}}^{\pm\alpha} \rangle = \frac{N}{4\pi} (g_{(0)\bar{\mu}\bar{\nu}} \pm \epsilon_{\bar{\mu}\bar{\nu}}) \mathcal{A}^{\pm\alpha\bar{\nu}}. \tag{5.11}$$

The vev of the stress energy tensor $T_{\bar{\mu}\bar{\nu}}$ is given by

$$\langle T_{\bar{\mu}\bar{\nu}} \rangle = \frac{N}{2\pi} \left(g_{(2)\bar{\mu}\bar{\nu}} + \frac{1}{2} R g_{(0)\bar{\mu}\bar{\nu}} + 8 \sum_r [\tilde{s}_i^{(r)1}]_1^2 g_{(0)\bar{\mu}\bar{\nu}} + \frac{1}{4} \left(\mathcal{A}_{(\bar{\mu}}^{+\alpha} \mathcal{A}_{\bar{\nu})}^{+\alpha} + \mathcal{A}_{(\bar{\mu}}^{-\alpha} \mathcal{A}_{\bar{\nu})}^{-\alpha} \right) \right) \tag{5.12}$$

where parentheses denote the symmetrized traceless combination of indices.

This summarizes the expressions for the vevs of chiral primaries with dimension two or less which were derived in [12]. Note that these operators correspond to supergravity fields which are at the bottom of each Kaluza-Klein tower. The supergravity solution of course also captures the vevs of operators dual to the other fields in each tower. Expressions for these vevs were not derived in [12], the obstruction being the non-linear terms: in general the vev of a dimension p operator will include contributions from terms involving up to p supergravity fields. Computing these in turn requires the field equations (along with gauge invariant combinations, KK reduction maps etc) up to p th order in the fluctuations.

Now (apart from the stress energy tensor) none of the operators whose vevs are given above is an SO(4) (R symmetry) singlet. For later purposes it will be useful to review

which other operators are SO(4) singlets. The computation of the linearized spectrum in [30] picks out the following as SO(4) singlets:

$$\tau^0 \equiv \frac{1}{12}\pi^0; \quad t^{(r)0} \equiv \frac{1}{4}\phi^{5(r)0}, \quad (5.13)$$

along with $\phi^{0i(r)}$ with $i = 1, \dots, 4$. Recall ψ^0 denotes the projection of the field ψ onto the degree zero harmonic. The fields $(\tau^0, t^{(r)0})$ are dual to operators of dimension four, whilst the fields $\phi^{0i(r)}$ are dual to dimension two (marginal) operators. The former lie in the same tower as $(\sigma^2, s^{(r)2})$ respectively, whilst the latter are in the same tower as $s^{(r)1}$. In total there are $(n_t + 1)$ SO(4) singlet irrelevant operators and $4n_t$ SO(4) singlet marginal operators, where $n_t = 5, 21$ for T^4 and $K3$ respectively.

Consider the SO(4) singlet marginal operators dual to the supergravity fields $\phi^{i(r)}$. These operators have been discussed previously in the context of marginal deformations of the CFT, see the review [32] and references therein. Suppose one introduces a free field realization for the T^4 theory, with bosonic and fermionic fields $(x_I^i(z), \psi_I^i(z))$ where $I = 1, \dots, N$. Then some of the marginal operators can be explicitly realized in the untwisted sector as bosonic bilinears

$$\partial x_I^i(z) \bar{\partial} x_I^j(\bar{z}); \quad (5.14)$$

there are sixteen such operators, in correspondence with sixteen of the supergravity fields. The remaining four marginal operators are realized in the twisted sector, and are associated with deformation from the orbifold point.

5.2 Application to the fuzzball solutions

The six-dimensional metric of (3.30) in the decoupling limit manifestly asymptotes to

$$ds^2 = \frac{r^2}{\sqrt{Q_1 Q_5}} (-dt^2 + dy^2) + \sqrt{Q_1 Q_5} \left(\frac{dr^2}{r^2} + d\Omega_3^2 \right). \quad (5.15)$$

where

$$Q_1 = \frac{Q_5}{L} \int_0^L dv (\dot{F}(v)^2 + \dot{\mathcal{F}}(v)^2 + \dot{\mathcal{F}}^{\alpha-}(v)^2). \quad (5.16)$$

Note that the vielbein (3.33) is asymptotically constant

$$V^o = \Omega_4^T \begin{pmatrix} I_2 & 0 & 0 & 0 \\ 0 & \sqrt{Q_1/Q_5} & 0 & 0 \\ 0 & 0 & \sqrt{Q_5/Q_1} & 0 \\ 0 & 0 & 0 & I_{22} \end{pmatrix} \Omega_4, \quad (5.17)$$

but it does not asymptote to the identity matrix. Thus one needs the constant SO(5, 21) transformation

$$V \rightarrow V(V^o)^{-1}, \quad G_3 \rightarrow V^o G_3. \quad (5.18)$$

to bring the background into the form assumed in (5.1).

The fields are expanded about the background values, by expanding the harmonic functions defining the solution in spherical harmonics as

$$\begin{aligned}
 H &= \frac{Q_5}{r^2} \sum_{k,I} \frac{f_{kI}^5 Y_k^I(\theta_3)}{r^k}, & K &= \frac{Q_1}{r^2} \sum_{k,I} \frac{f_{kI}^1 Y_k^I(\theta_3)}{r^k}, & (5.19) \\
 A_i &= \frac{Q_5}{r^2} \sum_{k \geq 1, I} \frac{(A_{kI})_i Y_k^I(\theta_3)}{r^k}, & \mathcal{A} &= \frac{\sqrt{Q_1 Q_5}}{r^2} \sum_{k \geq 1, I} \frac{(\mathcal{A}_{kI}) Y_k^I(\theta_3)}{r^k}, \\
 \mathcal{A}^{\alpha-} &= \frac{\sqrt{Q_1 Q_5}}{r^2} \sum_{k \geq 1, I} \frac{\mathcal{A}_{kI}^{\alpha-} Y_k^I(\theta_3)}{r^k}.
 \end{aligned}$$

The polar coordinates here are denoted by (r, θ_3) and $Y_k^I(\theta_3)$ are (normalized) spherical harmonics of degree k on S^3 with I labeling the degeneracy. Note that the restriction $k \geq 1$ in the last three lines is due to the vanishing zero mode, see [12]. As in [12], the coefficients in the expansion can be expressed as

$$\begin{aligned}
 f_{kI}^5 &= \frac{1}{L(k+1)} \int_0^L dv (C_{i_1 \dots i_k}^I F^{i_1} \dots F^{i_k}), & (5.20) \\
 f_{kI}^1 &= \frac{Q_5}{L(k+1)Q_1} \int_0^L dv \left(\dot{F}^2 + \dot{\mathcal{F}}^2 + (\dot{\mathcal{F}}^{\alpha-})^2 \right) C_{i_1 \dots i_k}^I F^{i_1} \dots F^{i_k}, \\
 (A_{kI})_i &= -\frac{1}{L(k+1)} \int_0^L dv \dot{F}_i C_{i_1 \dots i_k}^I F^{i_1} \dots F^{i_k}, \\
 (\mathcal{A}_{kI}) &= -\frac{\sqrt{Q_5}}{\sqrt{Q_1} L(k+1)} \int_0^L dv \dot{\mathcal{F}} C_{i_1 \dots i_k}^I F^{i_1} \dots F^{i_k}, \\
 \mathcal{A}_{kI}^{\alpha-} &= -\frac{\sqrt{Q_5}}{\sqrt{Q_1} L(k+1)} \int_0^L dv \dot{\mathcal{F}}^{\alpha-} C_{i_1 \dots i_k}^I F^{i_1} \dots F^{i_k}.
 \end{aligned}$$

Here the $C_{i_1 \dots i_k}^I$ are orthogonal symmetric traceless rank k tensors on \mathbb{R}^4 which are in one-to-one correspondence with the (normalized) spherical harmonics $Y_k^I(\theta_3)$ of degree k on S^3 . Fixing the center of mass of the whole system implies that

$$(f_{1i}^1 + f_{1i}^5) = 0. \quad (5.21)$$

The leading term in the asymptotic expansion of the transverse gauge field A_i can be written in terms of degree one vector harmonics as

$$A = \frac{Q_5}{r^2} (A_{1j})_i Y_1^j dY_1^i \equiv \frac{\sqrt{Q_1 Q_5}}{r^2} (a^{\alpha-} Y_1^{\alpha-} + a^{\alpha+} Y_1^{\alpha+}), \quad (5.22)$$

where $(Y_1^{\alpha-}, Y_1^{\alpha+})$ with $\alpha = 1, 2, 3$ form a basis for the $k = 1$ vector harmonics and we have defined

$$a^{\alpha\pm} = \frac{\sqrt{Q_5}}{\sqrt{Q_1}} \sum_{i>j} e_{\alpha ij}^{\pm} (A_{1j})_i, \quad (5.23)$$

where the spherical harmonic triple overlap $e_{\alpha ij}^{\pm}$ is defined in C.6. The dual field is given by

$$B = -\frac{\sqrt{Q_1 Q_5}}{r^2} (a^{\alpha-} Y_1^{\alpha-} - a^{\alpha+} Y_1^{\alpha+}). \quad (5.24)$$

Now given these asymptotic expansions of the harmonic functions one can proceed to expand all the supergravity fields, and extract the appropriate combinations required for computing the vevs defined in (5.9), (5.11) and (5.12). Since the details of the computation are very similar to those in [12], we will simply summarize the results as follows. Firstly the vevs of the stress energy tensor and of the R symmetry currents are the same as in [12], namely

$$\langle T_{\bar{\mu}\bar{\nu}} \rangle = 0; \tag{5.25}$$

$$\langle J^{\pm\alpha} \rangle = \pm \frac{N}{2\pi} a^{\alpha\pm} (dy \pm dt). \tag{5.26}$$

The vanishing of the stress energy tensor is as anticipated, since these solutions should be dual to R vacua. As in [12], however, the cancellation is very non-trivial. The vevs of the scalar operators dual to the fields $(s_I^{(6)k}, \sigma_I^k)$ are also unchanged from [12]:

$$\begin{aligned} \langle \mathcal{O}_{S_i^{(6)1}} \rangle &= \frac{N}{4\pi} (-4\sqrt{2}f_{1i}^5); \\ \langle \mathcal{O}_{S_I^{(6)2}} \rangle &= \frac{N}{4\pi} (\sqrt{6}(f_{2I}^1 - f_{2I}^5)); \\ \langle \mathcal{O}_{\Sigma_I^2} \rangle &= \frac{N}{4\pi} \sqrt{2}(- (f_{2I}^1 + f_{2I}^5) + 8a^{\alpha-} a^{\beta+} f_{I\alpha\beta}). \end{aligned} \tag{5.27}$$

The internal excitations of the new fuzzball solutions are therefore captured by the vevs of operators dual to the fields $s_I^{(r)k}$ with $r > 6$:

$$\begin{aligned} \langle \mathcal{O}_{S_i^{(5+n_t)1}} \rangle &= -\frac{N}{\pi} \sqrt{2}(\mathcal{A}_{1i}); & \langle \mathcal{O}_{S_i^{(6+\alpha_-)1}} \rangle &= \frac{N}{\pi} \sqrt{2}\mathcal{A}_{1i}^{\alpha_-}; \\ \langle \mathcal{O}_{S_I^{(5+n_t)2}} \rangle &= -\frac{N}{2\pi} \sqrt{6}(\mathcal{A}_{2I}); & \langle \mathcal{O}_{S_I^{(6+\alpha_-)2}} \rangle &= \frac{N}{2\pi} \sqrt{6}\mathcal{A}_{2I}^{\alpha_-}. \end{aligned} \tag{5.28}$$

Here $n_t = 5, 21$ for T^4 and $K3$ respectively, with $\alpha_- = 1, \dots, b^{2-}$ with $b^{2-} = 3, 19$ respectively. Thus each curve $(\mathcal{F}(v), \mathcal{F}^{\alpha_-}(v))$ induces corresponding vevs of operators associated with the middle cohomology of M^4 . Note the sign difference for the vevs of operators which are related to the distinguished harmonic function $\mathcal{F}(v)$.

6. Properties of fuzzball solutions

In this section we will discuss various properties of the fuzzball solutions, including the interpretation of the vevs computed in the previous section.

6.1 Dual field theory

Let us start by briefly reviewing aspects of the dual CFT and the ground states of the R sector; a more detailed review of the issues relevant here is contained in [12]. Consider the dual CFT at the orbifold point; there is a family of chiral primaries in the NS sector associated with the cohomology of the internal manifold, T^4 or $K3$. For our discussions only the chiral primaries associated with the even cohomology are relevant; let these be

labeled as $\mathcal{O}_n^{(p,q)}$ where n is the twist and (p, q) labels the associated cohomology class. The degeneracy of the operators associated with the $(1, 1)$ cohomology is $h^{1,1}$. The complete set of chiral primaries associated with the even cohomology is then built from products of the form

$$\prod_l (\mathcal{O}_{n_l}^{p_l, q_l})^{m_l}, \quad \sum_l n_l m_l = N, \quad (6.1)$$

where symmetrization over the N copies of the CFT is implicit. The correspondence between (scalar) supergravity fields and chiral primaries is ³

$$\begin{aligned} \sigma_n &\leftrightarrow \mathcal{O}_{(n-1)}^{(2,2)}, & n \geq 2; \\ s_n^{(6)} &\leftrightarrow \mathcal{O}_{(n+1)}^{(0,0)}, & s_n^{(6+\tilde{\alpha})} \leftrightarrow \mathcal{O}_{(n)\tilde{\alpha}}^{(1,1)}, & \tilde{\alpha} = 1, \dots, h^{1,1}, & n \geq 1. \end{aligned} \quad (6.2)$$

Spectral flow maps these chiral primaries in the NS sector to R ground states, where

$$\begin{aligned} h^R &= h^{NS} - j_3^{NS} + \frac{c}{24}; \\ j_3^R &= j_3^{NS} - \frac{c}{12}, \end{aligned} \quad (6.3)$$

where c is the central charge. Each of the operators in (6.1) is mapped by spectral flow to a (ground state) operator of definite R-charge

$$\begin{aligned} \prod_{l=1} (\mathcal{O}_{n_l}^{(p_l, q_l)})^{m_l} &\rightarrow \prod_{l=1} (\mathcal{O}_{n_l}^{R(p_l, q_l)})^{m_l}, \\ j_3^R &= \frac{1}{2} \sum_l (p_l - 1) m_l, & \bar{j}_3^R &= \frac{1}{2} \sum_l (q_l - 1) m_l. \end{aligned} \quad (6.4)$$

Note that R operators which are obtained from spectral flow of those associated with the $(1, 1)$ cohomology have zero R charge.

6.2 Correspondence between geometries and ground states

In [11, 12] we discussed the correspondence between fuzzball geometries characterized by a curve $F^i(v)$ and R ground states (6.4) with $(p_l, q_l) = 1 \pm 1$. The latter are related to chiral primaries in the NS sector built from the cohomology common to both T^4 and $K3$, namely the $(0, 0)$, $(2, 0)$, $(0, 2)$ and $(2, 2)$ cohomology.

The following proposal was made in [11, 12] for the precise correspondence between geometries and ground states; see also [33]. Given a curve $F^i(v)$ we construct the corresponding coherent state in the FP system and then find which Fock states in this coherent state have excitation number N_L equal to nw , where n is the momentum and w is the winding. Applying a map between FP oscillators and R operators then yields the superposition of R ground states that is proposed to be dual to the D1-D5 geometry.

This proposal can be straightforwardly extended to the new geometries, which are characterized by the curve $F^i(v)$ along with $h^{1,1}$ additional functions $(\mathcal{F}(v), \mathcal{F}^{\alpha-}(v))$. Consider

³As discussed in [12], the dictionary between $(\sigma_n, s_n^{(6)})$ and $(\mathcal{O}_{(n-1)}^{(2,2)}, \mathcal{O}_{(n+1)}^{(0,0)})$ may be more complicated, since their quantum numbers are indistinguishable, but this subtlety will not play a role here.

first the T^4 system, for which the four additional functions are $F^\rho(v)$. Then the eight functions $F^I(v) \equiv (F^i(v), F^\rho(v))$ can be expanded in harmonics as

$$F^I(v) = \sum_{n>0} \frac{1}{\sqrt{n}} (\alpha_n^I e^{-in\sigma^+} + (\alpha_n^I)^* e^{in\sigma^+}), \tag{6.5}$$

where $\sigma^+ = v/wR_9$. The corresponding coherent state in the FP system is

$$|F^I\rangle = \prod_{n,I} |\alpha_n^I\rangle, \tag{6.6}$$

where $|\alpha_n^I\rangle$ is a coherent state of the left moving oscillator \hat{a}_n^I , satisfying $\hat{a}_n^I |\alpha_n^I\rangle = \alpha_n^I |\alpha_n^I\rangle$. Contained in this coherent state are Fock states, such that

$$\prod (\hat{a}_{n_I}^I)^{m_I} |0\rangle, \quad N = \sum n_I m_I. \tag{6.7}$$

Now retain only the terms in the coherent state involving these Fock states, and map the FP oscillators to CFT R operators via the dictionary

$$\begin{aligned} \frac{1}{\sqrt{2}} (\hat{a}_n^1 \pm i\hat{a}_n^2) &\leftrightarrow \mathcal{O}_n^{R(\pm 1+1),(\pm 1+1)}; \\ \frac{1}{\sqrt{2}} (\hat{a}_n^3 \pm i\hat{a}_n^4) &\leftrightarrow \mathcal{O}_n^{R(\pm 1+1),(\mp 1+1)}; \\ \hat{a}_n^\rho &\leftrightarrow \mathcal{O}_{(\rho-4)n}^{R(1,1)}. \end{aligned} \tag{6.8}$$

The dictionary for the case of $K3$ is analogous. Here one has four curves $F^i(v)$ describing the transverse oscillations and twenty curves $\mathcal{F}^{\tilde{\alpha}}(v)$ describing the internal excitations. The oscillators associated with the former are mapped to operators associated with the universal cohomology as in (6.8) whilst the oscillators associated with the latter are mapped to operators associated with the $(1, 1)$ cohomology as

$$\hat{a}_n^{\tilde{\alpha}} \leftrightarrow \mathcal{O}_{\tilde{\alpha}n}^{R(1,1)}. \tag{6.9}$$

This completely defines the proposed superposition of R ground states to which a given geometry corresponds. Note that below we will suggest that a slight refinement of this dictionary may be necessary, taking into account that one of the internal curves is distinguished by the duality chain. For the distinguished curve the mapping may include a negative sign, namely $\hat{a}_n \leftrightarrow -\mathcal{O}_n^{R(1,1)}$; this mapping would explain the relative sign between the vevs found in (5.28) associated with the distinguished curve \mathcal{F} and the remaining curves \mathcal{F}^α respectively.

Note that there is a direct correspondence between the frequency of the harmonic on the curve and the twist label of the CFT operator. The latter is strictly positive, $n \geq 1$, and thus in the dictionary (6.8) there are no candidate CFT operators to correspond to winding modes of the curves $(\mathcal{F}(v), \mathcal{F}^{\alpha-}(v))$. In the case of T^4 such candidates might be provided by the additional chiral primaries associated with the extra T^4 in the target space of the sigma model, discussed in [34]. However the latter is related to the degeneracy of

the right-moving ground states in the dual F1-P system, rather than to winding modes. For $K3$ all chiral primaries have been included (except for the additional primaries which appear at specific points in the $K3$ moduli space). Thus one confirms that winding modes of the curves ($\mathcal{F}(v), \mathcal{F}^{\alpha-}(v)$) should not be included in constructing geometries dual to the R ground states. As discussed in appendix D these winding modes may describe geometric duals of states in deformations of the CFT.

6.3 Matching with the holographic vevs

In this section we will see how the general structure of the vevs given in (5.28) can be reproduced using the proposed dictionary. The holographic vevs take the form

$$\langle \mathcal{O}_{\tilde{\alpha}kI}^{(1,1)} \rangle \approx \frac{N\sqrt{Q_5}}{\sqrt{Q_1}L} \int_0^L dv \dot{\mathcal{F}}^{\tilde{\alpha}} C_{i_1 \dots i_k}^I F^{i_1} \dots F^{i_k}. \tag{6.10}$$

Thus the vevs of the operators $\mathcal{O}_{\tilde{\alpha}kI}^{(1,1)}$ are zero unless the curve $\mathcal{F}^{\tilde{\alpha}}(v)$ is non-vanishing and at least one of the $F^i(v)$ is non-vanishing. Moreover, the dimension one operators will not acquire a vev unless the transverse and internal curves have excitations with the same frequency. Analogous selection rules for frequencies of curve harmonics apply for the vevs of higher dimension operators.

These properties of the vevs follow directly from the proposed superpositions, along with selection rules for three point functions of chiral primaries. The superposition dual to a given set of curves is built from the R ground states

$$\mathcal{O}^{R\mathcal{I}}|0\rangle = \prod_l (\mathcal{O}_{n_l}^{R(p_l, q_l)})^{m_l}|0\rangle, \tag{6.11}$$

with $\sum_l n_l m_l = N$ and \mathcal{I} labeling the degeneracy of the ground states. So this superposition can be denoted abstractly as $|\Psi\rangle = \sum_{\mathcal{I}} a_{\mathcal{I}} \mathcal{O}^{R\mathcal{I}}|0\rangle$ with certain coefficients $a_{\mathcal{I}}$. In particular, if the curve $\mathcal{F}^{\tilde{\alpha}}(v) = 0$ the superposition does not contain any R ground states built from $\mathcal{O}_{\tilde{\alpha}n}^{R(1,1)}$ operators. Moreover, if there are no transverse excitations, the superposition will contain only states with zero R charge.

Now consider evaluating the vev of a dimension k operator $\mathcal{O}_{\tilde{\alpha}k}^{(1,1)}$ in such a superposition. This is determined by three point functions between this operator and the chiral primary operators occurring in the superposition. More explicitly, the operator vev is related to three point functions via

$$\langle \Psi_{NS} | \mathcal{O}_{\tilde{\alpha}k}^{(1,1)} | \Psi_{NS} \rangle = \sum_{\mathcal{I}, \mathcal{J}} a_{\mathcal{I}}^* a_{\mathcal{J}} \langle (\mathcal{O}^{\mathcal{I}})^\dagger(\infty) \mathcal{O}_{\tilde{\alpha}k}^{(1,1)}(\mu) (\mathcal{O}^{\mathcal{J}})(0) \rangle. \tag{6.12}$$

Here $\mathcal{O}^{\mathcal{I}}$ is the NS sector operator which flows to $\mathcal{O}^{R\mathcal{I}}$ in the R sector and $|\Psi_{NS}\rangle$ is the flow of the superposition back to the NS sector, namely $\sum_{\mathcal{I}} a_{\mathcal{I}} \mathcal{O}^{\mathcal{I}}|0\rangle$. The quantity μ is a mass scale. Note we are evaluating the relevant three point function in the NS sector, and have hence flowed the ground states back to NS sector chiral primaries. We would get the same answer by flowing the operator whose vev we wish to compute, $\mathcal{O}_{\tilde{\alpha}k}^{(1,1)}$, into the Ramond sector and computing the three point function there. Recall that the R charges

of these operators are related by the spectral flow formula (6.3) as $j_3^{NS} = j_3^R + \frac{1}{2}N$. In particular, NS sector chiral primaries built only from operators associated with the middle cohomology all have the same R charges, namely $\frac{1}{2}N$.

There are two basic selection rules for the three point functions (6.12). Firstly, as usual one has to impose conservation of the R charges. Secondly, a basic property of such three point functions is that they are only non-zero when the total number of operators $\mathcal{O}_{\tilde{\alpha}}^{(1,1)}$ with a given index $\tilde{\alpha}$ in the correlation function is even.⁴ From a supergravity perspective one can see this selection rule arising as follows. One computes n -point correlation functions using n -point couplings in the three dimensional supergravity action, with the latter following from the reduction of the ten-dimensional action on $S^3 \times M^4$. Since a (1, 1) form integrates to zero over M^4 , the three dimensional action only contains terms with an even number of fields $s^{\tilde{\alpha}}$ associated with a given (1, 1) cycle $\tilde{\alpha}$ on M^4 . Therefore non-zero n -point functions must contain an even number of operators $\mathcal{O}_{\tilde{\alpha}}^{(1,1)}$, and so do corresponding multi-particle 3-point functions obtained by taking coincident limits.

Expressed in terms of cohomology, allowed three point functions contain an even number of $(1,1)_{\tilde{\alpha}}$ cycles labeled by $\tilde{\alpha}$. Thus in single particle correlators one can have processes such as $\mathcal{O}^{(0,0)} + \mathcal{O}_{\tilde{\alpha}}^{(1,1)} \rightarrow \mathcal{O}_{\tilde{\alpha}}^{(1,1)}$ and $\mathcal{O}_{\tilde{\alpha}}^{(1,1)} + \mathcal{O}_{\tilde{\alpha}}^{(1,1)} \rightarrow \mathcal{O}^{(2,2)}$, but processes such as $\mathcal{O}^{(0,0)} + \mathcal{O}_{\tilde{\alpha}}^{(1,1)} \rightarrow \mathcal{O}^{(0,0)}$ which involve an odd number of $\tilde{\alpha}$ cycles are kinematically forbidden. This kinematical selection rule for (1, 1) cycle conservation immediately explains why the operator $\mathcal{O}_{\tilde{\alpha}k}^{(1,1)}$ can only acquire a vev when the curve $\mathcal{F}^{\tilde{\alpha}}(v)$ is non-vanishing: only then does the ground state superposition contain operators $\mathcal{O}_{\tilde{\alpha}}^{R(1,1)}$ such that the selection rule can be satisfied.

One can also easily see why the operator only acquires a vev if there are transverse excitations as well. All Ramond ground states associated with the middle cohomology have zero R charge, with the corresponding chiral primaries in the NS sector having the same charge $j_3^{NS} = \frac{1}{2}N$. Thus a superposition involving only $\mathcal{O}^{(1,1)}$ operators has a definite R charge, and a charged operator cannot acquire a vev. Including transverse excitations means that the superposition of Ramond ground states contains charged operators, associated with the universal cohomology, and does not have definite R charge. Therefore a charged operator can acquire a vev.

Thus, to summarize, the proposed map between curves and superpositions of R ground states indeed reproduces the principal features of the holographic vevs. Using basic selection rules for three point functions we have explained why the operators $\mathcal{O}_{\tilde{\alpha}k}^{(1,1)}$ acquire vevs only when the curve $\mathcal{F}^{\tilde{\alpha}}(v)$ is non-zero and when there are excitations in R^4 . We will see below that using reasonable assumptions for the three point functions we can also reproduce the selection rules for vevs relating to frequencies on the curves. Before discussing the general case, however, it will be instructive to consider a particular example.

6.4 A simple example

Consider a fuzzball geometry characterized by a circular curve in the transverse R^4 and

⁴Note that this selection rule was used for the computation of three point functions of single particle operators in the orbifold CFT in [35].

one additional internal curve, with only one harmonic of the same frequency:

$$F^1(v) = \frac{\mu A}{n} \cos\left(2\pi n \frac{v}{L}\right); \quad F^2(v) = \frac{\mu A}{n} \sin\left(2\pi n \frac{v}{L}\right); \quad \mathcal{F}(v) = \frac{\mu B}{n} \cos\left(2\pi n \frac{v}{L}\right), \quad (6.13)$$

where $\mu = \sqrt{Q_1 Q_5}/R$ and the D1-brane charge constraint (5.16) enforces

$$\left(A^2 + \frac{1}{2}B^2\right) = 1. \quad (6.14)$$

The corresponding dual superposition of R ground states is then given by

$$|\Psi\rangle = \sum_{l=0}^{N/n} C_l (\mathcal{O}_n^{R(2,2)})^l (\mathcal{O}_{1n}^{R(1,1)})^{\frac{N}{n}-l} |0\rangle, \quad (6.15)$$

$$C_l = \sqrt{\frac{(\frac{N}{n})!}{(\frac{N}{n}-l)! l!}} A^l \left(\frac{B}{\sqrt{2}}\right)^{\frac{N}{n}-l},$$

with the operators being orthonormal in the large N limit. In the case that either A or B are zero the superposition manifestly collapses to a single term. In the general case, this superposition gives the following for the expectation values of the R charges:

$$\begin{aligned} (\Psi|j_3^R|\Psi) &= (\Psi|\bar{j}_3^R|\Psi) = \frac{1}{2} \sum_{l=0}^{N/n} C_l^2 l; \\ &= \frac{N}{2n} \sum_{l=0}^{N/n-1} \frac{(\frac{N}{n}-1)!}{l!(\frac{N}{n}-l-1)!} A^{2(l+1)} \left(\frac{B}{\sqrt{2}}\right)^{2(\frac{N}{n}-(l+1))} = \frac{N}{2n} A^2. \end{aligned} \quad (6.16)$$

Evaluating (5.26) for (6.13) gives

$$\langle J^{\pm 3} \rangle = \frac{N}{2nR} A^2 (dy \pm dt), \quad (6.17)$$

and thus the integrated R charges defined in our conventions as

$$\langle j_3 \rangle = \frac{1}{2\pi} \int dy \langle J^{+3} \rangle; \quad \langle \bar{j}_3 \rangle = \frac{1}{2\pi} \int dy \langle J^{-3} \rangle, \quad (6.18)$$

agree with those of the superposition of R ground states.

The kinematical properties also match between the geometry and the proposed superposition. In particular, when $B \neq 0$ the $SO(2)$ symmetry in the 1-2 plane is broken: the harmonic functions (K, \mathcal{A}) depend explicitly on the angle ϕ in this plane. The asymptotic expansions of these functions involve charged harmonics, and therefore charged operators acquire vevs characterizing the symmetry breaking. More explicitly, the relevant terms in (5.20) are

$$\begin{aligned} f_{kI}^1 &\propto \int_0^L dv (A^2 + B^2 \sin^2(\frac{2\pi n v}{L})) C_{i_1 \dots i_k}^I F^{i_1} \dots F^{i_k}; \\ \mathcal{A}_{kI} &\propto \int_0^L dv B \sin(\frac{2\pi n v}{L}) C_{i_1 \dots i_k}^I F^{i_1} \dots F^{i_k}. \end{aligned} \quad (6.19)$$

Now the symmetric tensor of rank k and $\text{SO}(2)$ charge in the 1-2 plane of $\pm m$ behaves as

$$((F^1)^2 + (F^2)^2)^{k-m} (F^1 \pm iF^2)^m = \left(\frac{\mu A}{n}\right)^k e^{\pm 2\pi i n m \frac{v}{L}}. \quad (6.20)$$

Note that m is related to (j_3, \bar{j}_3) via $m = j_3 + \bar{j}_3$. Thus, when $B \neq 0$, harmonics in the expansion of f^1 with charges $|m| = 2$ are excited, and terms with $|m| = 1$ are excited in the expansion of \mathcal{A} . Following (6.10) the latter implies that the dimension k operators $\mathcal{O}_{1(km)}^{(1,1)}$ only acquire vevs when their $\text{SO}(2)$ charge m in the 1-2 plane is ± 1 . In particular using (5.28) the vevs of the dimension one operators are

$$\langle \mathcal{O}_{1(1\pm 1)}^{(1,1)} \rangle = \mp i \frac{N}{2\pi n} \mu AB, \quad (6.21)$$

where the normalized degree one symmetric traceless tensors are $\sqrt{2}(F^1 \pm iF^2)$.

These properties are implied by the superposition (6.15). The latter is a superposition of states with different R charge, and therefore it does break the $\text{SO}(2)$ symmetry, with the symmetry breaking being characterized by the vevs of charged operators. Moreover following (6.12) the vev of $\mathcal{O}_{1(km)}^{(1,1)}$ is given by

$$\sum_{l,l'} C_l^* C_{l'} \langle (\mathcal{O}_n^{(2,2)})^l (\mathcal{O}_{1n}^{(1,1)})^{\frac{N}{n}-l} | \mathcal{O}_{1(km)}^{(1,1)}(\mu) | (\mathcal{O}_n^{(2,2)})^{l'} (\mathcal{O}_{1n}^{(1,1)})^{\frac{N}{n}-l'} \rangle. \quad (6.22)$$

For the dimension one operators, charge conservation reduces this to

$$\sum_l C_{l\pm 1}^* C_l \langle (\mathcal{O}_n^{(2,2)})^{l\pm 1} (\mathcal{O}_{1n}^{(1,1)})^{\frac{N}{n}\mp 1-l} | \mathcal{O}_{1(1\pm 1)}^{(1,1)}(\mu) | (\mathcal{O}_n^{(2,2)})^l (\mathcal{O}_{1n}^{(1,1)})^{\frac{N}{n}-l} \rangle. \quad (6.23)$$

Thus there are contributions only from neighboring terms in the superposition. Computing the actual values of these vevs is beyond current technology: one would need to know three point functions for single and multiple particle chiral primaries at the conformal point. However, as in [12], the behavior of the vevs as functions of the curve radii (A, B) can be captured by remarkably simple approximations for the correlators, motivated by harmonic oscillators. Suppose one treats the operators as harmonic oscillators, with the operator $\mathcal{O}_{1(11)}^{(1,1)}$ destroying one $\mathcal{O}_{1n}^{(1,1)}$ and creating one $\mathcal{O}_n^{(2,2)}$. For harmonic oscillators such that $[\hat{a}, \hat{a}^\dagger] = 1$ the normalized state with p quanta is given by $|p\rangle = (\hat{a}^\dagger)^p / \sqrt{p!} |0\rangle$ and therefore $\hat{a}^\dagger |p\rangle = \sqrt{p+1} |p+1\rangle$. Using harmonic oscillator algebra for the operators gives

$$\langle (\mathcal{O}_n^{(2,2)})^{l+1} (\mathcal{O}_{1n}^{(1,1)})^{\frac{N}{n}-1-l} | \mathcal{O}_{1(11)}^{(1,1)}(\mu) | (\mathcal{O}_n^{(2,2)})^l (\mathcal{O}_{1n}^{(1,1)})^{\frac{N}{n}-l} \rangle \approx \mu \sqrt{\left(\frac{N}{n} - l\right)(l+1)}. \quad (6.24)$$

Then the corresponding vev in the superposition $|\Psi\rangle$ is

$$\langle \mathcal{O}_{1(11)}^{(1,1)} \rangle_\Psi = \mu \sum_{l=0}^{N/n-1} c_{l+1}^* c_l \sqrt{\left(\frac{N}{n} - l\right)(l+1)} = \mu \frac{N}{n} AB, \quad (6.25)$$

which has exactly the structure of (6.21). Given that such simple approximations (and factorizations) of the correlators reproduce the structure of the vevs so well, it would be

interesting to explore whether this relates to simplifications in the structure of the chiral ring in the large N limit.

Next consider the vevs of dimension k operators. Using charge conservation and $(1, 1)$ cycle conservation in (6.22) implies that only operators with m odd can acquire a vev. To reproduce the holographic result, that vevs are non-zero only when $m = \pm 1$, requires the assumption that only nearest neighbor terms in the superposition contribute to one point functions. This would follow from a stronger selection rule for $(1, 1)$ cycle conservation, that the number of $(1, 1)$ cycles in the in and out states differ by at most one. In particular, multi-particle processes such as $(\mathcal{O}_{\tilde{a}n}^{(1,1)})^3 + \mathcal{O}_{\tilde{\alpha}n}^{(1,1)} \rightarrow (\mathcal{O}_n^{(2,2)})^3$ would be forbidden. The selection rules for holographic vevs suggest that there is indeed such cycle conservation, and it would be interesting to explore this issue further.

Let us now return to the comment made below (6.9), that one may need to include a minus sign in the dictionary for the distinguished curve. Such a minus sign would introduce factors of $(-1)^{N/n-l}$ into the superposition (6.15), and thence an overall sign in the vevs of the associated operators $\mathcal{O}_{1(kI)}^{(1,1)}$. This would naturally account for the relative sign difference between the vevs associated with the distinguished curve and those associated with the remaining curves. It is not conclusive that one needs such a minus sign without knowing the exact three point functions and hence vevs. However such a sign change for oscillators associated with the direction distinguished by the duality would not be surprising. Recall that under T-duality of closed strings right moving oscillators associated with the duality direction switch sign, whilst the left moving oscillators and oscillators associated with orthogonal directions do not.

6.5 Selection rules for curve frequencies

Selection rules for charge and $(1, 1)$ cycles are sufficient to reproduce the general structure of the vevs. In the particular example discussed above, these rules also implied the selection rules for the curve frequencies: operators acquire vevs only when the transverse and internal curves have related frequencies.

Here we will note how, with reasonable assumptions, one can motivate the selection rules for frequencies in the general case. Consider the computation of the vev of a dimension one operator $\mathcal{O}_{\tilde{\alpha}1}^{(1,1)}$ for a general superposition $|\Psi\rangle$ using (6.12). Using the selection rules for charge and $(1, 1)$ cycles, the contributions to (6.12) involve only certain pairs of operators $(\mathcal{O}^{\mathcal{I}}, \mathcal{O}^{\mathcal{J}})$. Their $SO(2)$ charges must differ by $(\pm 1/2, \pm 1/2)$ and they must differ by an odd number of $\mathcal{O}_{\tilde{\alpha}}^{(1,1)}$ operators.

Now let us make the further assumption that there are contributions to (6.12) only from pairs of operators $(\mathcal{O}^{\mathcal{I}}, \mathcal{O}^{\mathcal{J}})$ which differ by only one term, the relevant operators taking the form

$$\mathcal{O}^{\mathcal{J}} = \mathcal{O}_n^{(p,q)} \mathcal{O}^{\tilde{\mathcal{J}}}, \tag{6.26}$$

with $\mathcal{O}^{\tilde{\mathcal{J}}}$ being the same for in and out states, but the single operator $\mathcal{O}_n^{(p,q)}$ differing between in and out states. Thus we are assuming that the relevant three point functions factorize, with the non-trivial part of the correlator arising from a single particle process.

This is indeed the structure of the three point functions arising in our example. Only nearest neighbor terms in the superposition contribute in the computation of the vev of the dimension one operator in (6.23). Moreover the $m = \pm 1$ charge selection rule for the vevs of higher dimension operators immediately follows from restricting to nearest neighbor terms in the three-point functions. Note further that this factorization structure is present in the orbifold CFT computation of the three point functions. The operator $\mathcal{O}_{\tilde{\alpha}1}^{(1,1)} \equiv \mathcal{O}_{\tilde{\alpha}1}^{(1,1)} I^{N-1}$ is the identity operator in $(N - 1)$ copies of the CFT and thus only acts non-trivially in one copy of the CFT.

Consider the case of the vev of the operator with $SO(2)$ charges $(1/2, 1/2)$; it would take the form

$$\sum_{\mathcal{I}, \mathcal{J}, \tilde{\mathcal{I}}} a_{\mathcal{I}}^* a_{\mathcal{J}} \mathcal{N}_{\tilde{\mathcal{I}}} \left(\langle (\mathcal{O}_n^{(2,2)})^\dagger(\infty) \mathcal{O}_{\tilde{\alpha}1}^{(1,1)}(\mu) (\mathcal{O}_{\tilde{\alpha}n}^{(1,1)}(0)) \rangle + \langle (\mathcal{O}_n^{(1,1)})^\dagger(\infty) \mathcal{O}_{\tilde{\alpha}1}^{(1,1)}(\mu) (\mathcal{O}_{\tilde{\alpha}n}^{(0,0)}(0)) \rangle \right), \tag{6.27}$$

where $\mathcal{N}_{\tilde{\mathcal{I}}}$ is the norm of $\mathcal{O}^{\tilde{\mathcal{I}}}$. Analogous expressions would hold for the dimension one operators with other charge assignments. Such a factorization would immediately explain the frequency selection rule found in the holographic vevs obtained from supergravity (6.10). The superposition contains operators of the form (6.26) with both $(p, q) = (1, 1)$ and $(p, q) \neq (1, 1)$ only when the internal curve and the transverse curves share a frequency. Extending these arguments to vevs of higher dimension operators would be straightforward, and would imply selection rules for curve frequencies.

6.6 Fuzzballs with no transverse excitations

Consider the case where the fuzzball geometry has only internal excitations, $F^i(v) = 0$. Then the corresponding dual superposition of ground states can involve only states built from the operators $\mathcal{O}_{\tilde{\alpha}n}^{R(1,1)}$. Any such state will be a zero eigenstate of both j_3^R and \bar{j}_3^R . Furthermore, such ground states associated with the middle cohomology account for a finite fraction of the entropy of the D1-D5 system. In the case of $K3$ the total entropy behaves as

$$S = 2\pi \sqrt{\frac{c}{6}}, \tag{6.28}$$

with $c = 24N$. The ground states associated with the middle cohomology account for a central charge $c = 20N$. In the case of T^4 the entropy behaves as (6.28) with $c = 12N$. The states associated with the universal cohomology account for $c = 4N$, the odd cohomology accounts for another $c = 4N$ and the middle cohomology accounts for the final $c = 4N$.

Now let us consider the properties of the corresponding fuzzball geometry. When there are no transverse excitations and no winding modes of the internal curves, the $SO(4)$ symmetry in R^4 is unbroken, and the defining harmonic functions (4.4) reduce to

$$H = 1 + \frac{Q_5}{r^2}; \quad K = \frac{Q_1}{r^2}; \tag{6.29}$$

with $A_i = 0$ and where Q_1 is defined in (5.16). The solutions manifestly all collapse to the standard (singular) D1-D5 solution and so, whilst one would need an exponential number

of geometries (upon quantization) to account for dual ground states build from operators associated with the middle cohomology, one has only one singular geometry. Therefore *the relevant fuzzball solutions are not visible in supergravity*: one needs to take into account higher order corrections.

One can understand this from several perspectives. Firstly, as discussed above, R ground states associated with the middle cohomology have zero R charge; they do not break the SO(4) symmetry. A geometry which is asymptotically $AdS_3 \times S^3$ for which the SO(4) symmetry is exact can be characterized by the vevs of SO(4) singlet operators. The only such operators in supergravity are the stress energy tensor, and the scalar operators listed in (5.13). Since the vev of the stress energy tensor must be zero for the D1-D5 ground states, the geometry would have to be distinguished by the vevs of the singlet operators given in (5.13).

Our results imply that these operators do not acquire vevs, and therefore within supergravity (without higher order corrections) geometries dual to different R ground states associated with the middle cohomology cannot be distinguished. The reason is the following. The SO(4) singlet operators dual to supergravity fields are related to chiral primaries by the action of supercharge raising operators; they are the top components of the multiplets. Thus these SO(4) singlet operators cannot acquire vevs in states built from the chiral primaries. SO(4) singlet operators associated with stringy excitations would be needed to characterize the different ground states.

A heuristic argument based on the supertube picture also indicates that geometries dual to these ground states are not to be found in the supergravity approximation. The geometries with transverse excitations in R^4 can be viewed as a bound state of D1-D5 branes, blown up by their angular momentum in the R^4 . Indeed, the characteristic size of the fuzzball geometry is directly related to this angular momentum. The simplest example, related to a circular supertube, is to take a geometry characterized by a circular curve; this is obtained by setting $B = 0$ in (6.13). The characteristic scale of the geometry is

$$r_c \sim g_s \mu / n, \tag{6.30}$$

where g_s is the string coupling and μ has dimensions of length, whilst the (dimensionless) angular momentum behaves as $j^{12} = N/n$, and thus $r_c \sim g_s \mu (j^{12}/N)$. Hence the size of the D1-D5 bound state increases linearly with the angular momentum. A general fuzzball geometry will of course not be as symmetric but nonetheless the characteristic scale averaged over the R^4 is still related to the total angular momentum. In our previous paper [12] we noted that fuzzball geometries dual to vacua for which the R charge is very small are not well described by supergravity. Here we have found that this implies that an exponential number of geometries dual to a finite fraction of the Ramond ground states, with strictly zero R charge, cannot be described at all in the supergravity approximation.

7. Implications for the fuzzball program

In this section we will consider the implications of our results for the fuzzball program, focusing in particular on whether one can find a set of smooth weakly curved supergravity

geometries which span the black hole microstates.

We have seen in the previous sections that the geometric duals of superpositions of R vacua with small or zero R charge are not well-described in supergravity. The natural basis for R ground states (6.4) uses states of definite R-charges, and it is therefore straightforward to work out the density of ground states with given R-charges, d_{N,j_3,\bar{j}_3} , with the total number of ground states being given by $d_N = \sum_{j_3,\bar{j}_3} d_{N,j_3,\bar{j}_3}$. This computation is discussed in appendix E with the resulting density in the large N limit being

$$d_{N,j^1,j^2} \cong \frac{1}{4(N+1-j)^{31/4}} \exp\left[\frac{2\pi(2N-j)}{\sqrt{N+1-j}}\right] \frac{1}{\cosh^2\left(\frac{\pi j^1}{\sqrt{N+1-j}}\right) \cosh^2\left(\frac{\pi j^2}{\sqrt{N+1-j}}\right)}, \quad (7.1)$$

where $j^1 = (j_3 + \bar{j}_3)$ and $j^2 = (j_3 - \bar{j}_3)$ and $j = |j^1| + |j^2|$. The key feature is that the number of states with zero R charge differs from the total number of R ground states given in (E.16) only by a polynomial factor:

$$d_{N,0,0} \cong d_N/N. \quad (7.2)$$

The geometries dual to such ground states are unlikely to be well-described in supergravity, and therefore the basis of black hole microstates labeled by R charges is not a good basis for the geometric duals. This argument reinforces the discussion of [12], where we showed in detail that the geometric duals of specific states (in this basis) must be characterized by very small vevs which cannot be reliably distinguished in supergravity; they are comparable in magnitude to higher order corrections.

The geometries that are smooth in supergravity correspond to specific superpositions of the R charge eigenstates, for which some vevs are atypically large. The natural basis for the field theory description of the microstates is thus *not* the natural basis for the geometric duals. This issue is likely to persist in other black hole systems. For example, the microstates of the D1-D5-P system are also most naturally described as (j_3, \bar{j}_3) eigenstates, with a relation analogous to (7.2) holding, so the number of states with zero R-charge is suppressed only polynomially compared to the total number of black hole microstates. Just as in the 2-charge system discussed here, the geometric duals are related to supertubes whose radii depend on the R-charges. States or superpositions of states which have small or zero R-charges are unlikely to be well-described by supergravity solutions. Thus a given smooth supergravity geometry should be described by a specific superposition of the black hole microstates. Identifying the specific superpositions for known 3-charge geometries is an open and important question.

The issue is whether there exist enough such geometries, well-described and distinguishable in supergravity, to span the entire set of black hole microstates. It seems unlikely that a basis exists which simultaneously satisfies all three requirements. Firstly, on general grounds microstates with small quantum numbers will not be well-described in supergravity. Even when considering superpositions that are well described by supergravity, to span the entire basis, one will have to include superpositions which can only be distinguished by these small vevs. I.e. in choosing a basis of geometries for which some vevs are sufficiently large for the supergravity description to be valid one will find that some of these geometries cannot be distinguished among themselves in supergravity.

We have already seen several examples of this problem in the 2-charge system. Let us parameterize the curves as

$$\begin{aligned}
 F^i(v) &= \mu \sum_n \left(\alpha_n^i e^{2\pi i n v/L} + (\alpha_n^i)^* e^{-2\pi i n v/L} \right); \\
 \mathcal{F}^{\tilde{\beta}}(v) &= \mu \sum_n \left(\alpha_n^{\tilde{\beta}} e^{2\pi i n v/L} + (\alpha_n^{\tilde{\beta}})^* e^{-2\pi i n v/L} \right),
 \end{aligned}
 \tag{7.3}$$

where $\mu = \sqrt{Q_1 Q_5}/R$ and $\tilde{\beta}$ runs from 1 to $h^{1,1}(M^4)$. The D1-brane charge constraint (4.7) limits the total amplitude of these curves as

$$\sum_n n^2 \left(|\alpha_n^i|^2 + |\alpha_n^{\tilde{\beta}}|^2 \right) = 1.
 \tag{7.4}$$

Thus in general increasing the amplitude in one mode, to make certain quantum numbers large, decreases the amplitudes in the others. Moreover, the amplitude in a given mode is bounded via $|\alpha_n|^2 \leq 1/n^2$, and is thus intrinsically very small for high frequency modes, which sample vacua with large twist labels in the CFT. Note also that the vevs of R-charges are given in terms of

$$j^{ij} = iN \sum_n n (\alpha_n^i (\alpha_n^j)^* - \alpha_n^j (\alpha_n^i)^*)
 \tag{7.5}$$

As we have seen, to be describable in supergravity, geometries must have transverse R^4 excitations, and thus some large R-charges, requiring $j^{ij} \gg 1$. Combining (7.5) and (7.4) one sees that this restricts the amplitudes of the internal excitations, and thus of the sampling of the black hole microstates associated with the middle cohomology of M^4 .

Another way to understand the limitations of supergravity is to go back to the F1-P system where the corresponding state is the coherent state $|\{\alpha_n^i\}, \{\alpha_m^{\tilde{\beta}}\}\rangle$. These states form a complete basis of states, so we know that there is an F1-P geometry corresponding to every 1/2 BPS state. However, only when all $\alpha_n^i, \alpha_m^{\tilde{\beta}}$ are large are the geometries well-described and distinguishable within supergravity. Indeed, the amplitudes $\alpha_n^i, \alpha_m^{\tilde{\beta}}$ are also the root mean deviations of the distribution around the mean (which is described by the classical curve), so only for large $\alpha_n^i, \alpha_m^{\tilde{\beta}}$ is the classical string that sources the supergravity solution a good approximation of the quantum state. Putting it differently, when some of the amplitudes are small the difference in the solutions for different amplitudes is comparable with the error in the solutions due to the approximation of the source by a classical string, so one cannot reliably distinguish them within this approximation.

If one could not find a basis of distinguishable supergravity geometries spanning the microstates, one might ask whether a sufficiently representative basis exists. That is, suppose one chooses a single representative of the indistinguishable geometries, and assigns a measure to this geometry. Then is the corresponding basis of weighted geometries sufficiently representative to obtain the black hole properties? In the 2-charge system, the now complete set of fuzzball geometries along with the precise mapping between these geometries and R vacua allows these questions to be addressed at a quantitative level and we hope to return to this issue elsewhere.

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A. Conventions

The following table summarises the indices used throughout the paper. In some cases an index is used more than once, with different meanings, in separate sections of the paper.

Index	Range	Usage
(m, n)	$0, \dots, 9$	10d sugra fields
(M, N)	$0, \dots, 5$	6d sugra fields
(μ, ν)	$0, 1, 2$	3d fields
(a, b)	$1, 2, 3$	S^3 indices
(i, j)	$1, 2, 3, 4$	R^4 indices
(ρ, σ)	$1, 2, 3, 4$	M^4 indices
$(\bar{\mu}, \bar{\nu})$	$0, 1$	2d fields
(α, β)	$1, 2, 3$	SU(2) vector index
(γ, δ)	$1, \dots, b^2$	$H_2(M^4)$
$(\tilde{\alpha}, \tilde{\beta})$	$1, \dots, h^{1,1}$	$H_{1,1}(M^4)$
(I, J)	$1, \dots, 8$	SO(8) vector
$((c), (d))$	$1, \dots, 16$	heterotic vector fields
$((a), (b))$	$1, \dots, 24$	SO(4, 20) vector
(A, B)	$1, \dots, 26$	SO(5, 21) vector
(m, n)	$1, \dots, 5$	SO(5) vector
(r, s)	$6, \dots, (n_t + 1)$	SO(n_t) vector

A.1 Field equations

The equations of motion for IIA supergravity are:

$$\begin{aligned}
 e^{-2\Phi} \left(R_{mn} + 2\nabla_m \nabla_n \Phi - \frac{1}{4} H_{mpq}^{(3)} H_n^{(3)pq} \right) - \frac{1}{2} F_{mp}^{(2)} F_n^{(2)p} \\
 - \frac{1}{2 \cdot 3!} F_{mpqr}^{(4)} F_n^{(4)pqr} + \frac{1}{4} G_{mn} \left(\frac{1}{2} (F^{(2)})^2 + \frac{1}{4!} (F^{(4)})^2 \right) = 0, \quad (\text{A.1}) \\
 4\nabla^2 \Phi - 4(\nabla \Phi)^2 + R - \frac{1}{12} (H^{(3)})^2 = 0, \\
 dH^{(3)} = 0, \quad dF^{(2)} = 0, \quad \nabla_m F^{(2)mn} - \frac{1}{6} H_{pqr}^{(3)} F^{(4)npqr} = 0,
 \end{aligned}$$

$$\begin{aligned} \nabla_m(e^{-2\Phi} H^{(3)mnp}) - \frac{1}{2} F_{qr}^{(2)} F^{(4)qrnp} - \frac{1}{2 \cdot (4!)^2} \epsilon^{npm_1 \dots m_4 n_1 \dots n_4} F_{m_1 \dots m_4}^{(4)} F_{n_1 \dots n_4}^{(4)} &= 0, \\ dF^{(4)} = H^{(3)} \wedge F^{(2)}, \quad \nabla_m F^{(4)mnpq} - \frac{1}{3! \cdot 4!} \epsilon^{npqm_1 \dots m_3 n_1 \dots n_4} H_{m_1 \dots m_3}^{(3)} F_{n_1 \dots n_4}^{(4)} &= 0. \end{aligned}$$

The corresponding equations for type IIB are:

$$\begin{aligned} e^{-2\Phi} \left(R_{mn} + 2\nabla_m \nabla_n \Phi - \frac{1}{4} H_{mpq}^{(3)} H_n^{(3)pq} \right) - \frac{1}{2} F_m^{(1)} F_n^{(1)} - \frac{1}{4} F_{mpq}^{(3)} F_n^{(3)pq} \\ - \frac{1}{4 \cdot 4!} F_{mpqrs}^{(5)} F_n^{(5)pqrs} + \frac{1}{4} G_{mn} \left((F^{(1)})^2 + \frac{1}{3!} (F^{(3)})^2 \right) &= 0, \\ 4\nabla^2 \Phi - 4(\nabla \Phi)^2 + R - \frac{1}{12} (H^{(3)})^2 &= 0, \\ dH^{(3)} = 0, \quad \nabla_m(e^{-2\Phi} H^{(3)mnp}) - F_m^{(1)} F^{(3)mnp} - \frac{1}{3!} F_{mqr}^{(3)} F^{(5)mqrnp} &= 0, \quad (\text{A.2}) \\ dF^{(1)} = 0, \quad \nabla_m F^{(1)m} + \frac{1}{6} H_{pqr}^{(3)} F^{(3)pqr} &= 0, \\ dF^{(3)} = H^{(3)} \wedge F^{(1)}, \quad \nabla_m F^{(3)mnp} + \frac{1}{6} H_{mqr}^{(3)} F^{(5)mqrnp} &= 0, \\ dF^{(5)} = d(*F^{(5)}) = H^{(3)} \wedge F^{(3)}, & \end{aligned}$$

where the Hodge dual of a p -form ω_p in d dimensions is given by

$$(*\omega_p)_{i_1 \dots i_{d-p}} = \frac{1}{p!} \epsilon_{i_1 \dots i_{d-p} j_1 \dots j_p} \omega_p^{j_1 \dots j_p}, \quad (\text{A.3})$$

with $\epsilon_{01 \dots d-1} = \sqrt{-g}$. The RR field strengths are defined as

$$F^{(p+1)} = dC^{(p)} - H^{(3)} \wedge C^{(p-2)}. \quad (\text{A.4})$$

The equations of motion for the heterotic theory are:

$$\begin{aligned} 4\nabla^2 \Phi - 4(\nabla \Phi)^2 + R - \frac{1}{12} (H^{(3)})^2 - \alpha' (F^{(c)})^2 &= 0, \\ \nabla_m \left(e^{-2\Phi} H^{(3)mnr} \right) &= 0, \\ R^{mn} + 2\nabla^m \nabla^n \Phi - \frac{1}{4} H^{(3)mrs} H_{rs}^{(3)n} - 2\alpha' F^{(c)mr} F_r^{(c)n} &= 0, \\ \nabla_m \left(e^{-2\Phi} F^{(c)mn} \right) + \frac{1}{2} e^{-2\Phi} H^{(3)mrs} F_{rs}^{(c)} &= 0. \end{aligned}$$

$F_{mn}^{(c)}$ with $(c) = 1, \dots, 16$ are the field strengths of Abelian gauge fields $V_m^{(c)}$; we consider here only supergravity backgrounds with Abelian gauge fields. This restriction means that the gauge field part of the Chern-Simons form in H_3 ,

$$H^{(3)} = dB^{(2)} - 2\alpha' \omega_3(V) + \dots, \quad (\text{A.5})$$

does not play a role in the supergravity solutions, nor does the Lorentz Chern-Simons term denoted by the ellipses.

A.2 Duality rules

The T-duality rules for RR fields were derived in [36] by reducing type IIA and type IIB supergravities on a circle and relating the respective RR potentials in the 9-dimensional theory. However, for calculations involving magnetic sources, it is more convenient to work with T-duality rules for RR field strengths, since potentials can only be defined locally. In the following we will rederive the T-duality rules in terms of RR field strengths.

It is slightly easier although not necessary to use the democratic formalism of IIA and IIB supergravity introduced in [16]. In this formalism one includes p -form field strengths for $p > 5$ with Hodge dualities relating higher and lower-form field strengths being imposed in the field equations. This formalism is natural when both magnetic and electric sources are present; moreover there is no need for Chern-Simons terms in the field equations. The RR part of the (pseudo)-action is simply

$$S_{RR} = -\frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \sum_q \frac{1}{4q!} (F^{(q)})^2, \quad (\text{A.6})$$

where $q = 2, 4, 6, 8$ is even in the IIA case and $q = 1, 3, 5, 7, 9$ is odd in the IIB case. The field strengths are defined as $F^{(q)} = dC^{(q-1)} - H^{(3)} \wedge C^{(q-3)}$ for $q \geq 3$ and $F_q = dC^{(q-1)}$ for $q < 3$. The Hodge duality relation between higher and lower form field strengths in our conventions is

$$*F^{(q)} = (-1)^{\lfloor \frac{q}{2} \rfloor} F^{(10-q)}, \quad (\text{A.7})$$

where $\lfloor n \rfloor$ denotes the largest integer less or equal to n .

Now to compactify on a circle the ten-dimensional metric can be parameterized as

$$ds^2 = e^{2\psi} (dy - A_\mu dx^\mu)^2 + \hat{g}_{\mu\nu} dx^\mu dx^\nu, \quad (\text{A.8})$$

where y denotes the compact direction, and 9-dimensional quantities will be denoted as hatted. An economic way to derive the T-duality rules for the field strengths is the following. Choose the vielbein to be

$$e^{\underline{y}} = e^\psi (dy - A_\mu dx^\mu); \quad e^{\underline{\mu}} = \hat{e}^{\underline{\mu}}, \quad (\text{A.9})$$

where $\underline{\mu}$ denotes a tangent space index, and $\hat{e}^{\underline{\mu}}$ is the 9-dimensional vielbein. Now reduce the field strengths (in the tangent frame) as

$$\hat{F}_{\underline{\mu}_1 \dots \underline{\mu}_q}^{(q)} = F_{\underline{\mu}_1 \dots \underline{\mu}_q}^{(q)}, \quad \hat{F}_{\underline{\mu}_1 \dots \underline{\mu}_{q-1}}^{(q-1)} = F_{\underline{\mu}_1 \dots \underline{\mu}_{q-1} \underline{y}}^{(q)}. \quad (\text{A.10})$$

The corresponding 9-dimensional action for the field strengths is given by

$$S_{RR} = -\frac{2\pi R}{2\kappa_{10}^2} \int d^9x \sqrt{-\hat{g}} \sum_{q=1}^9 \frac{1}{4q!} e^\psi \hat{F}_q^2. \quad (\text{A.11})$$

Since $\psi_{IIA} = -\psi_{IIB}$ under T-duality, one can read from this action the transformation rules for field strengths in 10d:

$$\begin{aligned} \tilde{F}_{\underline{\mu}_1 \dots \underline{\mu}_q \underline{y}}^{(q+1)} &= e^\psi F_{\underline{\mu}_1 \dots \underline{\mu}_q}^{(q)}, \\ \tilde{F}_{\underline{\mu}_1 \dots \underline{\mu}_{q+1}}^{(q+1)} &= e^\psi F_{\underline{\mu}_1 \dots \underline{\mu}_{q+1} \underline{y}}^{(q+2)}. \end{aligned} \quad (\text{A.12})$$

Here q even defines IIB fields in terms of IIA fields and q odd defines IIA in terms of IIB. Note that the field strengths on both sides are in the tangent frame. Given the T-duality rules for NSNS fields

$$\begin{aligned} e^{\tilde{\psi}} &= e^{-\psi}, & \tilde{A}_\mu &= B_{y\mu}^{(2)}, & \tilde{B}_{ym}^{(2)} &= A_m, \\ \tilde{B}_{mn}^{(2)} &= B_{mn}^{(2)} + 2A_{[m}B_{n]y}^{(2)}, & \tilde{\Phi} &= \Phi - \psi, \end{aligned} \tag{A.13}$$

with the metric g_{mn} invariant, one can easily convert (A.12) back into

$$\begin{aligned} F_{m_1\dots m_q}^{(q)} &= F_{m_1\dots m_q y}^{(q+1)} - q(-1)^q B_{y[m_1} F_{m_2\dots m_q]}^{(q-1)} + q(q-1)B_{y[m_1} A_{m_2} F_{m_3\dots m_q]y}^{(q-1)} \\ F_{m_1\dots m_{q-1}y}^{(q)} &= F_{m_1\dots m_{q-1}}^{(q-1)} - (q-1)(-1)^q A_{[m_1} F_{m_2\dots m_{q-1]y}^{(q-1)}. \end{aligned} \tag{A.14}$$

Strictly speaking, this gives the duality rules in the democratic formalism. However we can obtain the usual rules by simply dropping the ($p > 5$)-form field strengths as long as we make sure to self-dualise $F^{(5)}$ in each IIB solution.

The S duality rules for type IIB are

$$\begin{aligned} \tilde{\tau} &= -\frac{1}{\tau}, & \tilde{B}^{(2)} &= C^{(2)}, & \tilde{C}^{(2)} &= -B^{(2)}, \\ \tilde{F}^{(5)} &= F^{(5)}, & \tilde{G}_{mn} &= |\tau|G_{mn}, \end{aligned} \tag{A.15}$$

where $\tau = C^{(0)} + ie^{-\Phi}$.

B. Reduction of type IIB solutions on $K3$

The reduction of type IIB on $K3$ is very similar to the reduction of type IIA, which was discussed in some detail in [37]. In the following we will use the reduction of the NS-NS sector fields given in [37], and derive the reduction of the type IIB RR fields. Let us first review the reduction of the NS-NS sector. Starting from the ten-dimensional action

$$S_{NS} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-\hat{g}} \left(e^{-2\hat{\Phi}} \left(\hat{R} + 4(\partial\hat{\Phi})^2 - \frac{1}{12}\hat{H}_3^2 \right) \right), \tag{B.1}$$

where ten-dimensional fields are denoted by hats, the corresponding six-dimensional field equations can be derived from the action [37]

$$S = \frac{1}{2\kappa_6^2} \int d^6x \sqrt{-g} e^{-2\Phi} \left(R + 4(\partial\Phi)^2 - \frac{1}{12}H_3^2 + \frac{1}{8}\text{tr}(\partial M^{-1}\partial M) \right), \tag{B.2}$$

where the six-dimensional fields are defined as follows. Firstly the 10-dimensional 2-form potential is reduced as

$$\hat{B}^{(2)}(x, y) = B_2(x) + b^\gamma(x)\omega_2^\gamma(y), \tag{B.3}$$

where (x, y) are six-dimensional and $K3$ coordinates respectively and the two forms ω_2^γ with $\gamma = 1, \dots, 22$ span the cohomology $H^2(K3, \mathbb{R})$. The 2-forms ω_2^γ transform under an $O(3, 19)$ symmetry, with a metric defined by the 22-dimensional intersection matrix

$$d_{\gamma\delta} = \frac{1}{(2\pi)^4 V} \int_{K3} \omega_2^\gamma \wedge \omega_2^\delta, \tag{B.4}$$

where $(2\pi)^4 V$ is the volume of $K3$. A natural choice for $d_{\gamma\delta}$ is

$$d_{\gamma\delta} = \begin{pmatrix} I_3 & 0 \\ 0 & -I_{19} \end{pmatrix}, \quad (\text{B.5})$$

corresponding to a diagonal basis for the 3 self-dual and 19 anti-self dual two forms of $K3$. Furthermore, there is a matrix D^δ_γ defined by the action of the Hodge operator

$$*_4^{K3} \omega_2^\gamma = \omega_2^\delta D^\delta_\gamma, \quad (\text{B.6})$$

which is dependent on the $K3$ metric and satisfies

$$D^\gamma_\delta D^\delta_\epsilon = \delta^\gamma_\epsilon, \quad D^\epsilon_\delta d_{\epsilon\zeta} D^\zeta_\gamma = d_{\delta\zeta}. \quad (\text{B.7})$$

The $\text{SO}(4, 20)$ matrix of scalars $M_{(a)(b)}^{-1}$ was derived in [37] to be

$$M^{-1} = \Omega_2^T \begin{pmatrix} e^{-\rho} + b^\gamma b^\delta d_{\gamma\epsilon} D^\epsilon_\delta + \frac{1}{4} e^\rho b^4 & \frac{1}{2} e^\rho b^2 & \frac{1}{2} e^\rho b^2 b^\gamma d_{\gamma\delta} + b^\gamma d_{\gamma\epsilon} D^\epsilon_\delta \\ \frac{1}{2} e^\rho b^2 & e^\rho & e^\rho b^\gamma d_{\gamma\delta} \\ \frac{1}{2} e^\rho b^2 b^\gamma d_{\gamma\delta} + b^\gamma d_{\gamma\epsilon} D^\epsilon_\delta & e^\rho b^\gamma d_{\gamma\delta} & e^\rho b^\epsilon d_{\epsilon\gamma} b^\zeta d_{\zeta\delta} + d_{\gamma\epsilon} D^\epsilon_\delta \end{pmatrix} \Omega_2, \quad (\text{B.8})$$

with $b^2 \equiv b^\gamma b^\delta d_{\gamma\delta}$. Here ρ is the breathing mode of $K3$, $e^{-\rho} = \frac{1}{(2\pi)^4 V} \int_{K3} *_4 1$. The six-dimensional dilaton is related to the 10-dimensional dilaton via $\Phi = \hat{\Phi} + \rho/2$.

The dimensional reduction of the NS sector makes manifest only an $\text{SO}(4, 20)$ subgroup of the full $\text{SO}(5, 21)$ symmetry. Including the reduction of the RR sector should thus give the equations of motion following from the six-dimensional string frame action, which for IIB was given in (3.23)

$$S = \frac{1}{2\kappa_6^2} \int d^6 x \sqrt{-g} \left\{ e^{-2\Phi} \left(R + 4(\partial\Phi)^2 + \frac{1}{8} \text{tr}(\partial M^{-1} \partial M) \right) + \frac{1}{2} \partial^{(a)} M_{(a)(b)}^{-1} \partial^{(b)} - \frac{1}{3} G_{MNP}^A \mathcal{M}_{AB}^{-1} G^{BMNP} \right\},$$

and in which only an $\text{SO}(4, 20)$ subgroup of the total $\text{SO}(5, 21)$ symmetry is manifest; recall that \mathcal{M}_{AB}^{-1} here is an $\text{SO}(5, 21)$ matrix, with $M_{(a)(b)}^{-1}$ being $\text{SO}(4, 20)$. Note that the six-dimensional coupling is related to the ten-dimensional coupling via $(2\pi)^4 V (2\kappa_6^2) = 2\kappa_{10}^2$, where $(2\pi)^4 V$ is the volume of $K3$.

Following the same steps as [37] the RR potentials can be reduced as

$$\begin{aligned} \hat{C}^{(0)}(x, y) &= C_0(x), & \hat{C}^{(2)}(x, y) &= C_2(x) + c_{(0,2)}^\gamma(x) \omega_2^\gamma(y), \\ \hat{C}^{(4)}(x, y) &= C_4(x) + c_{(2,4)}^\gamma(x) \wedge \omega_2^\gamma(y) + c_{(0,4)}(x) (e^\rho *_4 1)(y), \end{aligned} \quad (\text{B.9})$$

where $*_{K3}$ denotes the Hodge dual in the $K3$ metric and the corresponding field strengths are

$$\begin{aligned}
 \hat{F}^{(1)}(x, y) &= F_1(x), \\
 \hat{F}^{(3)}(x, y) &= dC_2(x) - C_0(x)H_3(x) + \left(dc_{(0,2)}^\gamma(x) - C_0(x)db^\gamma(x) \right) \omega_2(y) \equiv F_3 + K_1^\gamma \wedge \omega_2^\gamma, \\
 \hat{H}^{(3)}(x, y) &= dB_2(x) + db^\gamma(x) \wedge \omega_2^\gamma(y) \equiv H_3 + db^\gamma \wedge \omega_2^\gamma, \\
 \hat{F}^{(5)}(x, y) &= dC_4(x) - C_2(x) \wedge H_3(x) + \left(dc_{(2,4)}^\gamma(x) - C_2(x)db^\gamma(x) - c_{(0,2)}^\gamma(x)H_3(x) \right) \wedge \omega_2^\gamma(y) \\
 &\quad + \left(dc_{(0,4)}(x) - c_{0,2}^\gamma(x)db^\delta(x)d_{\gamma\delta} \right) \wedge (e^\rho(x) *_{K3} 1)(y) \\
 &\equiv F_5 + K_3^\gamma \wedge \omega_2^\gamma + \tilde{F}_1 \wedge e^\rho *_{K3} 1.
 \end{aligned} \tag{B.10}$$

The reduction of the potentials thus gives two three form field strengths H_3 and F_3 , 3 self-dual and 19 anti-self dual three form field strengths K_3^γ and 46 scalars b^γ , $c_{(0,2)}^\gamma$, $c_{(0,4)}$ and C_0 . After splitting the three forms H_3 and F_3 into their self-dual and anti-self-dual parts, we obtain 5 self-dual and 21 anti-self dual tensors in total, as described in [38].

It is then straightforward to obtain the map relating six and ten-dimensional fields by inserting the expressions (B.9) and (B.10) into the ten-dimensional field equations (A.2). The additional RR scalars are contained in

$$l^{(a)} = \Omega_2^T \begin{pmatrix} C_0 \\ \tilde{c}_{(0,4)} \\ \tilde{c}_{(0,2)}^\gamma \end{pmatrix}, \tag{B.11}$$

with Ω_2 defined in the appendix B.2 and the shifted fields defined as

$$\begin{aligned}
 \tilde{c}_{(0,2)}^\gamma &= c_{(0,2)}^\gamma - C_0 b^\gamma, \\
 \tilde{c}_{(0,4)} &= c_{(0,4)} - b^\gamma c_{(0,2)}^\delta d_{\gamma\delta} + \frac{1}{2} b^2 C_0.
 \end{aligned} \tag{B.12}$$

The fields Φ , $l^{(a)}$ and the $SO(4, 20)$ matrix M^{-1} given in (B.8) can be recombined into the $SO(5, 21)$ matrix $\mathcal{M}^{-1} = V^T V$, with the latter conveniently expressed in terms of the vielbein

$$V = \Omega_4^T \begin{pmatrix} e^{-\Phi} & 0 & 0 & 0 & 0 \\ -e^\Phi (C_0 c_{(0,4)} - \frac{1}{2} c_{(0,2)}^2) & e^\Phi & -e^\Phi \tilde{c}_{(0,4)} & -e^\Phi C_0 & e^\Phi \tilde{c}_{(0,2)}^\gamma d_{\gamma\delta} \\ e^{-\rho/2} C_0 & 0 & e^{-\rho/2} & 0 & 0 \\ e^{\rho/2} c_{(0,4)} & 0 & \frac{1}{2} e^{\rho/2} b^2 & e^{\rho/2} & e^{\rho/2} b^\gamma d_{\gamma\delta} \\ \tilde{V}_{\delta\gamma} c_{(0,2)}^\gamma & 0 & \tilde{V}_{\delta\gamma} b^\gamma & 0 & \tilde{V}_{\gamma\delta} \end{pmatrix} \Omega_4. \tag{B.13}$$

Here the $SO(3, 19)$ vielbein $\tilde{V}_{\alpha\beta}$ is defined by $d_{\alpha\beta} D^\beta_\gamma = \tilde{V}_{\alpha\beta} \tilde{V}_{\beta\gamma}$, $c_{(0,2)}^2 \equiv c_{(0,2)}^\gamma c_{(0,2)}^\delta d_{\gamma\delta}$ and the matrix Ω_4 is defined in the appendix B.2. The six-dimensional tensor fields are related

to the ten-dimensional fields as

$$\begin{aligned}
 H_3^1 &= \frac{e^{-\Phi}}{4}(1 + *_6)H_3, & H_3^{\alpha_+ + 1} &= -\frac{1}{\sqrt{8}}(\tilde{V}K_3)^{\alpha_+}, \\
 H_3^5 &= -\frac{e^{-\rho/2}}{4}(1 + *_6)F_3, & H_3^6 &= -\frac{e^{-\rho/2}}{4}(1 - *_6)F_3, \\
 H_3^{\alpha_- + 3} &= -\frac{1}{\sqrt{8}}(\tilde{V}K_3)^{\alpha_-}, & H_3^{26} &= \frac{e^{-\Phi}}{4}(1 - *_6)H_3.
 \end{aligned}
 \tag{B.14}$$

Here $\alpha_+ = 1, 2, 3$ and $\alpha_- = 4, \dots, 22$, labeling the self dual and anti-self dual forms respectively. Note that using formulas (B.13) and (B.14) to lift a six-dimensional solution to ten dimensions requires a specific choice of six-dimensional vielbein.

The solutions we find have $D_\delta^\gamma = d_{\gamma\delta}$; this implies the identity

$$(\omega_2^{\alpha_-})_{\rho\sigma}(\omega_2^{\beta_-})_{\tau}{}^\sigma = \frac{1}{2}g_{\rho\tau}\delta^{\alpha_-\beta_-},
 \tag{B.15}$$

where (ρ, τ) are $K3$ coordinates and $g_{\rho\tau}$ is the $K3$ metric. As discussed in [39], a choice of D^γ_ϵ fixes the complex structure completely and implies $(\omega_2^\gamma)_{\rho\sigma}(\omega_2^\delta)^{\rho\sigma} = D^\epsilon_\delta d_{\gamma\epsilon}$. Varying this identity with respect to the metric results in (B.15).

B.1 S-duality in 6 dimensions

Given the map between 10-dimensional and 6-dimensional fields, we can now obtain the action of S-duality on 6-dimensional fields as part of the $SO(5, 21)$ symmetry:

$$G_3 \rightarrow O_S G_3, \quad \mathcal{M}^{-1} \rightarrow O_S \mathcal{M}^{-1} O_S^T,
 \tag{B.16}$$

where

$$(O_S)_{ij} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_3 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (O_S)_{rs} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{19} & 0 \\ -1 & 0 & 0 \end{pmatrix},
 \tag{B.17}$$

Moreover one can perform an $SO(5) \times SO(21)$ transformation to bring the vielbein of the S-dual solution back to the form used by the 10-dimensional lift. Including this transformation, H_3 and V transform as

$$H_3 \rightarrow O_G H_3, \quad V \rightarrow O_G V O_S^T,
 \tag{B.18}$$

with

$$(O_G)_{ij} = \frac{1}{|\tau|} \begin{pmatrix} C_0 & 0 & -e^{\hat{\Phi}} \\ 0 & I_3 & 0 \\ e^{\hat{\Phi}} & 0 & C_0 \end{pmatrix}, \quad (O_G)_{rs} = \frac{1}{|\tau|} \begin{pmatrix} C_0 & 0 & -e^{\hat{\Phi}} \\ 0 & I_{19} & 0 \\ e^{\hat{\Phi}} & 0 & C_0 \end{pmatrix},
 \tag{B.19}$$

where $\tau = C_0 + ie^{-\hat{\Phi}}$, $\hat{\Phi} = \Phi - \rho/2$ is the 10-dimensional dilaton and the fields C_0 and $e^{\hat{\Phi}}$ are the original ones taken before the S-duality.

B.2 Basis change matrices

In defining six-dimensional supergravities there are implicit choices of constant $SO(p, q)$ matrices. When discussing the compactification from the ten to six dimensions, the most convenient choices for these matrices are certain off-diagonal forms, see for example [15, 17, 19–22]. When one is interested in specific solutions of the six-dimensional supergravity equations, such as $AdS_3 \times S^3$ solutions, and deriving the spectrum in such backgrounds, it is rather more convenient to use diagonal choices for these matrices, see for example [30, 31]. In this paper we both compactify from ten to six dimensions, and expand six-dimensional solutions about a given background. We therefore find it most convenient to use diagonal choices for the constant matrices. To use previous results on compactification and T-duality, we need to apply certain similarity transformations. For the most part these may be implicitly written in terms of basis change matrices, so that compactification and duality formulas remain as simple as possible. Thus let us define matrices Ω_1 and Ω_2 for $SO(4, 20)$, and Ω_3 and Ω_4 for $SO(5, 21)$ via:

$$\begin{aligned} \Omega_1^T \begin{pmatrix} v^\rho \\ w^\rho \\ x^{(c)} \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{2}}(v^\rho - w^\rho) \\ \frac{1}{\sqrt{2}}(v^\rho + w^\rho) \\ x^{(c)} \end{pmatrix}, & \Omega_3^T \begin{pmatrix} v \\ w \\ x^{(a)} \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{2}}(v - w) \\ x^{(a)} \\ \frac{1}{\sqrt{2}}(v + w) \end{pmatrix}, & (B.20) \\ \Omega_2^T \begin{pmatrix} v \\ w \\ x^\alpha \\ y^{\alpha-} \end{pmatrix} &= \begin{pmatrix} x^\alpha \\ \frac{1}{\sqrt{2}}(v - w) \\ \frac{1}{\sqrt{2}}(v + w) \\ y^{\alpha-} \end{pmatrix}, & \Omega_4^T \begin{pmatrix} v_1 \\ w_1 \\ v_2 \\ w_2 \\ x^\alpha \\ y^{\alpha-} \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{2}}(v_1 - w_1) \\ x^\alpha \\ \frac{1}{\sqrt{2}}(v_2 - w_2) \\ \frac{1}{\sqrt{2}}(v_2 + w_2) \\ y^{\alpha-} \\ \frac{1}{\sqrt{2}}(v_1 + w_1) \end{pmatrix}, \end{aligned}$$

where $\rho = 1, \dots, 4$, $(c) = 1, \dots, 16$, $(a) = 1, \dots, 24$, $\alpha = 1, 2, 3$ and $\alpha_- = 1, \dots, 19$. These satisfy the conditions:

$$\begin{aligned} \Omega_1 \begin{pmatrix} 0 & -I_4 & 0 \\ -I_4 & 0 & 0 \\ 0 & 0 & -I_{16} \end{pmatrix} \Omega_1^T &= \begin{pmatrix} I_4 & 0 \\ 0 & -I_{20} \end{pmatrix}, & (B.21) \\ \Omega_2 \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & -I_{19} \end{pmatrix} \Omega_2^T &= \begin{pmatrix} I_4 & 0 \\ 0 & -I_{20} \end{pmatrix}, \\ \Omega_3 \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & I_4 & 0 \\ 0 & 0 & -I_{20} \end{pmatrix} \Omega_3^T &= \begin{pmatrix} I_5 & 0 \\ 0 & -I_{21} \end{pmatrix}, \\ \Omega_4 \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_1 & 0 & 0 \\ 0 & 0 & I_3 & 0 \\ 0 & 0 & 0 & -I_{19} \end{pmatrix} \Omega_4^T &= \begin{pmatrix} I_5 & 0 \\ 0 & -I_{21} \end{pmatrix}. \end{aligned}$$

Here σ_1 is the Pauli matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

C. Properties of spherical harmonics

Scalar, vector and tensor spherical harmonics satisfy the following equations

$$\begin{aligned} \square Y^I &= -\Lambda_k Y^I, \\ \square Y_a^{I_v} &= (1 - \Lambda_k) Y_a^{I_v}, & D^a Y_a^{I_v} &= 0, \\ \square Y_{(ab)}^{I_t} &= (2 - \Lambda_k) Y_{(ab)}^{I_t}, & D^a Y_{k(ab)}^{I_t} &= 0, \end{aligned} \quad (\text{C.1})$$

where $\Lambda_k = k(k+2)$ and the tensor harmonic is traceless. It will often be useful to explicitly indicate the degree k of the harmonic; we will do this by an additional subscript k , e.g. degree k spherical harmonics will also be denoted by Y_k^I , etc. \square denotes the d'Alembertian along the three sphere. The vector spherical harmonics are the direct sum of two irreducible representations of $SU(2)_L \times SU(2)_R$ which are characterized by

$$\epsilon_{abc} D^b Y^{cI_v \pm} = \pm(k+1) Y_a^{I_v \pm} \equiv \lambda_k Y_a^{I_v \pm}. \quad (\text{C.2})$$

The degeneracy of the degree k representation is

$$d_{k,\epsilon} = (k+1)^2 - \epsilon, \quad (\text{C.3})$$

where $\epsilon = 0, 1, 2$ respectively for scalar, vector and tensor harmonics. For degree one vector harmonics I_v is an adjoint index of $SU(2)$ and will be denoted by α . We use normalized spherical harmonics such that

$$\int Y^{I_1} Y^{J_1} = \Omega_3 \delta^{I_1 J_1}; \quad \int Y^{aI_v} Y_a^{J_v} = \Omega_3 \delta^{I_v J_v}; \quad \int Y^{(ab)I_t} Y_{(ab)}^{J_t} = \Omega_3 \delta^{I_t J_t}, \quad (\text{C.4})$$

where $\Omega_3 = 2\pi^2$ is the volume of a unit 3-sphere. We define the following triple integrals as

$$\int Y^I Y^J Y^K = \Omega_3 a_{IJK}; \quad (\text{C.5})$$

$$\int (Y_1^{\alpha \pm})^a Y_1^j D_a Y_1^i = \Omega_3 e_{\alpha ij}^{\pm}; \quad (\text{C.6})$$

D. Interpretation of winding modes

In the fundamental string supergravity solutions (2.1) the null curves describing the motion of the string along a torus direction x^ρ (whose periodicity is $2\pi R_\rho$) could have winding modes such that $F_\rho(v) = w_\rho R_\rho v / R_y$, with w_ρ integral. Consider now the correspondence with quantum string states. Such winding modes are not consistent with both supersymmetry and momentum and winding quantization for a string propagating in flat space, with

no B field. Recall that the zero modes of a worldsheet compact boson field can be written as

$$X(\sigma^+, \sigma^-) = x + \frac{1}{2} \left(\alpha' \frac{p}{R} + nR \right) \sigma^+ + \frac{1}{2} \left(\alpha' \frac{p}{R} - nR \right) \sigma^- \equiv x + \tilde{w} \sigma^+ + w \sigma^-, \quad (\text{D.1})$$

where R is the radius and (p, n) are the quantized momentum and winding respectively; note that we define $\sigma^\pm = (\tau \pm \sigma)$. BPS left-moving states with no right-moving excitations have $w = 0$ and hence $\alpha' p = -nR^2$. However the latter condition has no solutions at generic radius and so states with winding along the torus directions cannot be BPS. Therefore winding modes should not be included to describe the F1-P states and corresponding dual D1-D5 ground states of interest here.

Now consider switching on constant $B_{\rho v}^{(2)} \equiv b_\rho$ on the worldsheet. The constant B field shifts the momentum charges, and thus there are BPS left-moving states with winding around the torus directions. To be more precise, following the discussion of [12], one can describe a string with left-moving excitations using a null lightcone gauge. The relevant terms in the worldsheet fields are then

$$V = w^v \sigma^-; \quad U = w^u \sigma^- + \tilde{w}^u \sigma^+ + \sum_n \frac{1}{\sqrt{n}} a_n^- e^{-in\sigma^-}; \quad (\text{D.2})$$

$$X^I = \delta_{I\rho} w^I \sigma^- + \sum_n \frac{1}{\sqrt{n}} a_n^I e^{-in\sigma^-},$$

where winding modes are included only along torus directions, labeled by ρ . The L_0 constraint implies

$$w^v w^u = (w^\rho)^2 + 2 \sum_{n>0} |n| a_n^I a_{-n}^I \equiv (w^v)^2 |\partial_V X^I|_0^2, \quad (\text{D.3})$$

where $|A|_0$ denotes the projection onto the zero mode. The momentum and winding charges are given by

$$P^m = \frac{1}{4\pi} \int d\sigma (\partial_\tau X^m + B_{mn}^{(2)} \partial_\sigma X^n); \quad W^m = \frac{1}{2\pi} \int d\sigma \partial_\sigma X^m, \quad (\text{D.4})$$

respectively, where $\alpha' = 2$. Requiring no winding in the time direction and no momentum along the x^ρ directions imposes $\tilde{w}^u = w^u + w^v$ and $w^\rho = b_\rho w^v$. The conserved momentum and winding charges are then

$$P^m = \frac{1}{2} w^v ((1 + |\partial_V X^I|_0^2 + b_\rho^2), (|\partial_V X^I|_0^2 - b_\rho^2), 0); \quad W^m = w^v (0, 1, 0, b_\rho). \quad (\text{D.5})$$

Note that the integral quantized momentum charge p_y along the y direction is therefore

$$p_y = R_y (w^u - (w^v)^{-1} (w^\rho)^2). \quad (\text{D.6})$$

Now consider the solitonic string supergravity solution (2.1) with defining curves $F^I(v)$ where $F^\rho(v) = b_\rho v + \bar{F}^\rho(v)$, with $\bar{F}^\rho(v)$ having no zero mode. The ADM charges of this solitonic string were computed in [15], and are given by

$$P_{\text{ADM}}^m = kQ ((1 + |\partial_v F^I|_0^2), |\partial_v F^I|_0^2, 0, b_\rho), \quad (\text{D.7})$$

where the effective Newton constant is $k = \Omega_3 L_y / 2\kappa_6^2$. When $b_\rho = 0$ these charges match the worldsheet charges (D.5) provided that $w^v = 2kQ$ as in [15] but when $b_\rho \neq 0$ they do not quite agree with the worldsheet charges. The reason is that in the supergravity solution $B_{\rho v}^{(2)}$ approaches zero at infinity, but to match with the constant $B_{\rho v}^{(2)}$ background on the worldsheet, $B_{\rho v}^{(2)}$ should approach b_ρ at infinity. This can be achieved via a constant gauge transformation $A_\rho \rightarrow A_\rho - b_\rho$, combined with a coordinate shift $u \rightarrow u + 2b_\rho x^\rho$. The ADM charges of this shifted background indeed exactly match the worldsheet charges (D.5). The harmonic functions A_ρ then take the form

$$A_\rho = -b_\rho H - \frac{Q}{L_y} \int_0^{L_y} dv \frac{\partial_v \bar{F}^\rho}{|x - F|^2}, \tag{D.8}$$

where in the latter expression $|x - F|^2$ denotes $\sum_i (x^i - F^i(v))^2$; the harmonic function has been smeared over the T^4 and the y circle. Note that when $F^i(v) = 0$ the supergravity solution collapses to

$$\begin{aligned} ds^2 &= H^{-1} dv(-du + K dv) + dx^I dx_I; & K &= \left(1 + \frac{Q|\partial_v F^\rho|_0^2}{r^2}\right), \\ e^{-2\Phi} &= H \equiv \left(1 + \frac{Q}{r^2}\right); & B_{uv}^{(2)} &= \frac{1}{2}(H^{-1} - 1); & B_{v\rho}^{(2)} &= -b_\rho. \end{aligned} \tag{D.9}$$

This is the naive SO(4) invariant F1-P solution, with an additional constant B field. Finally let us note that one can similarly switch on winding modes for the curves $q^{(c)}(v)$ characterizing the charge waves in the heterotic solution (3.1) by including constant $A_v^{(c)}$ on the worldsheet.

Now let us consider solutions in the D1-D5 system, and the interpretation of including winding modes of the internal curves. In particular, it is interesting to note that the general SO(4) invariant solutions include harmonic functions

$$\mathcal{A} = a_o + \frac{a}{r^2}; \quad \mathcal{A}^{\alpha-} = a_o^{\alpha-} + \frac{a^{\alpha-}}{r^2}, \tag{D.10}$$

in addition to the harmonic functions (H, K) given in (6.29). The non-constant terms in these harmonic functions are related to the winding modes of the internal curves, with the quantities $a^{\tilde{\alpha}} = (a, a^{\alpha-})$ being given by

$$a = -\frac{Q_5}{L} \int_0^L dv \dot{\mathcal{F}}(v); \quad a^{\alpha-} = -\frac{Q_5}{L} \int_0^L dv \dot{\mathcal{F}}^{\alpha-}(v). \tag{D.11}$$

Following the duality chain, these constants are given by $a^{\tilde{\alpha}} = -Q_5 b^{\tilde{\alpha}}$ where for the T^4 case $b^{\tilde{\alpha}} \equiv B_{\rho v}^{(2)} = b_\rho$ and for the $K3$ case $b^{\tilde{\alpha}} \equiv (B_{\rho v}^{(2)} = b_\rho, A_v^{(c)} = b^{(c)})$. The constant terms $(a_o, a_o^{\alpha-})$ are related to the boundary conditions at asymptotically flat infinity, as we will discuss below.

When these functions $(\mathcal{A}, \mathcal{A}^{\alpha-})$ are non-zero, the geometry generically differs from the naive D1-D5 geometry. The functions (f_1, \tilde{f}_1) appearing in the metric behave as

$$\begin{aligned}\tilde{f}_1 &= 1 + \frac{Q_1}{r^2} - \left(1 + \frac{Q_5}{r^2}\right)^{-1} \left(\left(a_o + \frac{a}{r^2}\right)^2 + \left(a_o^{\alpha-} + \frac{a^{\alpha-}}{r^2}\right)^2 \right) \\ f_1 &= 1 + \frac{Q_1}{r^2} - \left(1 + \frac{Q_5}{r^2}\right)^{-1} \left(\left(a_o^{\alpha-} + \frac{a^{\alpha-}}{r^2}\right)^2 \right).\end{aligned}\quad (\text{D.12})$$

In the decoupling limit these functions become

$$\tilde{f}_1 \rightarrow r^{-2}(Q_1 - Q_5^{-1}(a^2 + a^{\alpha-}a^{\alpha-})) \equiv \frac{\tilde{q}_1}{r^2}; \quad f_1 \rightarrow r^{-2}(Q_1 - Q_5^{-1}(a^{\alpha-}a^{\alpha-})) \equiv \frac{q_1}{r^2}, \quad (\text{D.13})$$

and thus $(a_o, a_o^{\alpha-})$ drop out. Note that \tilde{q}_1 corresponds to the conserved momentum charge in the F1-P system (D.6). Substituting the decoupling region functions into (4.1), one finds that the near horizon region of the solution is $AdS_3 \times S^3 \times M^4$, supported by both $F^{(3)}$ and $H^{(3)}$ flux:

$$\begin{aligned}ds^2 &= \frac{r^2\sqrt{q_1}}{\tilde{q}_1\sqrt{Q_5}}(-dt^2 + dy^2) + \sqrt{q_1Q_5}\left(\frac{dr^2}{r^2} + d\Omega_3^2\right) + \frac{\sqrt{q_1}}{\sqrt{Q_5}}ds_{M^4}^2; \\ e^{2\Phi} &= \frac{q_1^2}{Q_5\tilde{q}_1}, \quad F_{tyr}^{(3)} = -\frac{2r}{\tilde{q}_1}, \quad F_{\Omega_3}^{(3)} = 2q_1^{-1}\tilde{q}_1Q_5; \\ H_{tyr}^{(3)} &= 2aQ_5^{-1}\tilde{q}_1^{-1}r, \quad H_{\Omega_3}^{(3)} = -2a.\end{aligned}\quad (\text{D.14})$$

The field strengths $F^{(1)}$ and $F^{(5)}$ vanish, but there are non-vanishing potentials:

$$\begin{aligned}B_{\rho\sigma}^{(2)} &= \sqrt{2}Q_5^{-1}a^{\alpha-}\omega_{\rho\sigma}^{\alpha-}, \quad C^{(0)} = -q_1^{-1}a, \quad C_{\rho\sigma\tau\pi}^{(4)} = Q_5^{-1}a\epsilon_{\rho\sigma\tau\pi}; \\ C_{ty\alpha\beta}^{(4)} &= a(1 + \tilde{q}_1^{-1}r^2)\epsilon_{\alpha\beta}, \quad C_{\alpha\beta\rho\sigma}^{(4)} = 2\sqrt{2}\epsilon_{\alpha\beta}a^{\alpha-}\omega_{\rho\sigma}^{\alpha-}, \quad C_{ty\rho\sigma}^{(4)} = \sqrt{2}Q_5^{-1}a^{\alpha-}\omega_{\rho\sigma}^{\alpha-},\end{aligned}\quad (\text{D.15})$$

where ϵ is a 2-form such that $d\epsilon$ is the volume form of the unit 3-sphere. The conserved charges therefore include Chern-Simons terms; using the equations of motion (A.2) one finds that they are given by

$$\begin{aligned}D5 : Q_5 &= \frac{1}{2} \int_{S^3} (F^{(3)} + H^{(3)}C^{(0)}); \\ D1 : \tilde{q}_1 &= \frac{1}{2} \int_{S^3 \times M^4} (*F^{(3)} + H^{(3)} \wedge C^{(4)}); \\ D3 : a^{\alpha-} &= \frac{1}{2\sqrt{2}} \int_{S^3 \times \omega^{\alpha-}} B^{(2)} \wedge (F^{(3)} + H^{(3)}C^{(0)}); \\ NS5 : a &= -\frac{1}{2} \int_{S^3} H^{(3)},\end{aligned}\quad (\text{D.16})$$

where we drop terms which do not contribute to the charges. The curvature radius of the $AdS_3 \times S^3$ is $l = (q_1Q_5)^{1/4}$, and the three-dimensional Newton constant is

$$\frac{1}{G_3} = \frac{8\pi V_4 \Omega_3 \tilde{q}_1}{\kappa_{10}^2 q_1} (q_1Q_5)^{3/4}, \quad (\text{D.17})$$

with the volume of M^4 being $(2\pi)^4 V$ and $2\kappa_{10}^2 = (2\pi)^7 (\alpha')^4$. Then using [40, 41] the central charge of the dual CFT is

$$c = \frac{3l}{2G_3} = 6 \frac{V}{(\alpha')^4} \tilde{q}_1 Q_5 \equiv 6\tilde{n}_1 n_5 \tag{D.18}$$

where the integral charges (\tilde{n}_1, n_5) are given by

$$Q_5 = \alpha' n_5; \quad \tilde{q}_1 = \frac{(\alpha')^3 \tilde{n}_1}{V}. \tag{D.19}$$

Now consider the relation between this system and the F1-P system discussed previously. The conserved charges here are $(Q_5, \tilde{q}_1, a, a^{\alpha-})$, which correspond to the winding, momentum along the y circle and winding along the internal manifold in the original system. The fact that $(a, a^{\alpha-})$ measure NS5-brane and D3-brane charges in the final system is consistent with the duality chains from the F1-P systems: applying the standard duality rules along the chains given in (2.6), (2.7) and (3.4), one indeed finds that the original winding charges become NS5-brane and D3-brane charges.

Finally let us comment on the constant terms in the harmonic functions, $(a_o, a_o^{\alpha-})$. These clearly determine the behavior of the solution at asymptotically flat infinity: the B field and RR potentials at infinity depend on them. Now consider how these constant terms can be described in the CFT. In the context of the pure D1-D5 system it was noted in [12] that (infinitesimal) constant terms in the harmonic functions (f_1, f_5) can be reinstated by making (infinitesimal) irrelevant deformations of the CFT by $SO(4)$ singlet operators. See also [42] for a related discussion in the context of the AdS_5/CFT_4 correspondence. It seems probable that a similar interpretation would hold here: the $(n_t - 1)$ parameters $(a_o, a_o^{\alpha-})$ (where $n_t = 5, 21$ for T^4 and $K3$ respectively) would be related to the parameters of deformations of the CFT by irrelevant $SO(4)$ singlet operators. In total taking into account these $(n_t - 1)$ zero modes, plus the two constant terms in the (f_1, f_5) harmonic functions, one gets $(n_t + 1)$ parameters. This agrees exactly with the count of the number of irrelevant $SO(4)$ singlet operators.⁵ How to describe these deformations in the field theory beyond the infinitesimal level is not known, however.

E. Density of ground states with fixed R charges

In this appendix we will derive an asymptotic formula for the number of R ground states with given R charges. Our derivation follows closely that of [43] for the density of fundamental string states with a given mass and angular momentum. In fact, we will consider the case of $K3$, so the relevant counting is precisely that of the density of left moving heterotic string states with a given excitation level N and (commuting) angular momenta (j^{12}, j^{34}) in the transverse R^4 . For this purpose we can consider the following Hamiltonian

$$H = \sum_{n=1}^{\infty} \left(\sum_{(a)=1}^{24} \alpha_{-n}^{(a)} \alpha_n^{(a)} \right) + \lambda_1 j^1 + \lambda_2 j^2, \tag{E.1}$$

⁵Such deformations may also be related to the attractor flow of moduli; this idea is currently being developed by Kyriakos Papadodimas and collaborators.

where (λ_1, λ_2) are Lagrange multipliers and

$$j^1 = j^{12} = -i \sum_{n=1}^{\infty} n^{-1} (\alpha_{-n}^1 \alpha_n^2 - \alpha_{-n}^2 \alpha_n^1); \quad j^2 = j^{34} = -i \sum_{n=1}^{\infty} n^{-1} (\alpha_{-n}^3 \alpha_n^4 - \alpha_{-n}^4 \alpha_n^3). \quad (\text{E.2})$$

Here the oscillators satisfy the standard commutation relations, namely $[\alpha_n^{(a)}, \alpha_m^{(b)}] = n \delta_{n+m} \delta^{(a)(b)}$. In [43] the partition function was computed in the case $\lambda_2 = 0$, and thus the partition function of interest here can be computed by generalizing their results. The first step is to diagonalize the Hamiltonian by introducing combinations

$$a_n^{12} = \frac{1}{\sqrt{2n}} (\alpha_n^1 + i \alpha_n^2); \quad b_n^{12} = \frac{1}{\sqrt{2n}} (\alpha_n^1 - i \alpha_n^2) \quad (\text{E.3})$$

and analogously (a_n^{34}, b_n^{34}) . Then the Hamiltonian takes the form

$$H = \sum_{n=1}^{\infty} \left(\sum_{(a)=5}^{24} \alpha_{-n}^{(a)} \alpha_n^{(a)} + (n - \lambda_1) (a_n^{12})^\dagger a_n^{12} + (n + \lambda_1) (b_n^{12})^\dagger b_n^{12} \right. \\ \left. + (n - \lambda_2) (a_n^{34})^\dagger a_n^{34} + (n + \lambda_2) (b_n^{34})^\dagger b_n^{34} \right) \quad (\text{E.4})$$

The partition function $Z = \text{Tr}(e^{-\beta H})$ is then

$$Z = \prod_{n=1}^{\infty} [(1 - w^n)^{-20} (1 - c_1 w^n)^{-1} (1 - c_1^{-1} w^n)^{-1} (1 - c_2 w^n)^{-1} (1 - c_2^{-1} w^n)^{-1}] \quad (\text{E.5})$$

with $w = e^{-\beta}$ and $c_1 = e^{\beta \lambda_1}$, $c_2 = e^{\beta \lambda_2}$. To estimate the asymptotic density of states, one as usual expresses the partition function in terms of modular functions and then uses the modular transformation properties. Here one needs the Jacobi theta function

$$\theta_1(z|\tau) = 2f(q^2)q^{1/4} \sin(\pi z) \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos(2\pi z) + q^{4n}), \quad (\text{E.6})$$

with

$$f(q^2) = \prod_{n=1}^{\infty} (1 - q^{2n}), \quad q = e^{i\pi\tau}, \quad (\text{E.7})$$

and the modular transformation property

$$\theta_1\left(-\frac{z}{\tau} \middle| -\frac{1}{\tau}\right) = e^{i\pi/4} \sqrt{\tau} e^{i\pi z^2/\tau} \theta_1(z|\tau) \quad (\text{E.8})$$

Rewriting the partition function in terms of the modular functions, applying this modular transformation and then taking the high temperature limit results in

$$Z(\beta, \lambda_1, \lambda_2) = C \beta^{12} e^{4\pi^2/\beta} \frac{\lambda_1 \lambda_2}{\sin(\pi \lambda_1) \sin(\pi \lambda_2)}, \quad (\text{E.9})$$

with C a constant. From this expression one can extract the density of states with level N and angular momenta (j^1, j^2) by expanding

$$Z(w, k_1, k_2) = \sum_{N,j} d_{N,j^1,j^2} w^N e^{ik_1 j^1 + ik_2 j^2}, \quad (\text{E.10})$$

where $k_1 = -i\beta\lambda_1$ and $k_2 = -i\beta\lambda_2$, and projecting out the d_{N,j^1,j^2} . Integrating over (k_1, k_2) can be done exactly, since

$$\int_{-\infty}^{\infty} dk e^{iky} \frac{k}{\sinh(\pi k/\beta)} = \frac{1}{2} \beta^2 \frac{1}{\cosh^2(\beta y/2)}, \quad (\text{E.11})$$

resulting in the following contour integral over a circle around $w = 0$ for d_{N,j^1,j^2} :

$$d_{N,j^1,j^2} = C' \oint \frac{dw}{w^{N+1}} \beta^{14} e^{4\pi^2/\beta} \frac{1}{\cosh^2(\beta j^1/2) \cosh^2(\beta j^2/2)}. \quad (\text{E.12})$$

Assuming N is large the integral can be approximated by a saddle point evaluation, with the saddle point defined by the solution of

$$\frac{4\pi^2}{\beta^2} = N + 1 - j^1 \tanh\left(\frac{1}{2} j^1 \beta\right) - j^2 \tanh\left(\frac{1}{2} j^2 \beta\right). \quad (\text{E.13})$$

For small angular momenta, which is the case of primary interest here, the solution is $\beta \cong 2\pi/\sqrt{N+1}$. For $(|j^1|, |j^2|) = \mathcal{O}(N)$ the stationary point is at

$$\beta \cong \frac{2\pi}{\sqrt{N+1 - |j^1| - |j^2|}}. \quad (\text{E.14})$$

Note that $|j^1| + |j^2| \leq N$. This latter stationary point is equally applicable to small angular momenta, and thus one can write the asymptotic density of states as

$$d_{N,j^1,j^2} \cong \frac{1}{4(N+1-j)^{31/4}} \exp\left[\frac{2\pi(2N-j)}{\sqrt{N+1-j}}\right] \frac{1}{\cosh^2\left(\frac{\pi j^1}{\sqrt{N+1-j}}\right) \cosh^2\left(\frac{\pi j^2}{\sqrt{N+1-j}}\right)}, \quad (\text{E.15})$$

where $j = |j^1| + |j^2|$. The constant of proportionality is fixed by the state with $j^1 = N$, $j^2 = 0$ being unique. Note that the commuting generators (j_3, \bar{j}_3) of $(\text{SU}(2)_L, \text{SU}(2)_R)$ respectively are related to the rotations in the 1-2 and 3-4 planes via $j_3 = \frac{1}{2}(j^1 + j^2)$ and $\bar{j}_3 = \frac{1}{2}(j^1 - j^2)$. The total number of states at level N is

$$d_N \cong \frac{1}{N^{27/4}} \exp(4\pi\sqrt{N}), \quad (\text{E.16})$$

and thus the density of states with zero angular momenta differs from the total number of states only by a factor of $1/N$; the exponential growth with N is the same.

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